VISCOSITY SOLUTION OF LINEAR REGULATOR QUADRATIC FOR DEGENERATE DIFFUSIONS

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The paper studied a linear regulator quadratic control problem for degenerate Hamilton-Jacobi-Bellman (HJB) equation. We showed the existence of viscosity properties and established a unique viscosity solution of the degenerate HJB equation associated with this problem by the technique of viscosity solutions.

1. Introduction

We are concerned with the linear regulator quadratic control problem to minimize the expected cost with discount factor $\alpha > 0$:

$$J(c) = E\left[\int_0^\infty e^{-\beta t}\left\{h(x_t) + |c_t|^2\right\}dt\right], \tag{1.1}$$

over $c \in \mathscr{A}$ subject to the degenerate stochastic differential equation

$$dx_t = [Ax_t + c_t]dt + \sigma x_t dw_t, \quad x_0 = x \in \mathbb{R}, \ t \geq 0, \tag{1.2}$$

for nonzero constants A, $\sigma \neq 0$, and a continuous function h on $\mathbb{R}$, where w_t is a one-dimensinal standard Brownian motion on a complete probability space $(\Omega, \mathscr{F}, P)$ endowed with the natural filtration $\mathscr{F}_t$ generated by $\sigma(w_s, s \leq t)$, and $\mathscr{A}$ denotes the class of all $\mathscr{F}_t$−progressively measurable processes $c = (c_t)$ with $J(c) < \infty$.

This kind of stochastic control problem has been studied by many authors [3, 7] for nondegenerate diffusions to (1.1) and (1.2). We also assume that h satisfies the following properties:

$$h(x) \geq 0: \text{ convex}; \tag{1.3}$$

$$\text{There exists } C > 0 \quad \text{such that } h(x) \leq C(1 + |x|^n), \ x \in \mathbb{R}, \tag{1.4}$$

for some constant $C > 0$, $n \geq 0$. We refer to [5] for the quadratic case of degenerate diffusions related to Riccati equations in case of $h(x) = Cx^2$ and $n = 2$ with infinite horizon.

The purpose of this paper is to show the existence of a viscosity solution of u of the associated Hamilton-Jacobi-Bellman (in short, HJB) equation of the form:

$$-\beta u + \frac{1}{2}\sigma^2 x^2 u'' + Axu' + \min_{r \in \mathbb{R}} (r^2 + ru') + h(x) = 0 \quad \text{in } \mathbb{R}. \tag{1.5}$$

Our method consists in finding a unique viscosity solution for u of (1.5) following Bardi and Capuzzo-Dolcetta [2], Crandall et al. [4], Fleming and Soner [7] through the limit of the solution $v = v_L$, $L > 0$, to the HJB equation

$$-\beta v + \frac{1}{2}\sigma^2 x^2 v'' + Axv' + \min_{|r| \leq L} (r^2 + rv') + h(x) = 0 \quad \text{in } \mathbb{R}, \tag{1.6}$$

as $L \to \infty$. To show the existence of the viscosity solution v_L, we assume that h has the following property: there exists $C_\rho > 0$, for any $\rho > 0$, such that

$$|h(x) - h(y)| \leq C_\rho |x - y|^n + \rho(1 + |x|^n + |y|^n), \quad \forall x, y \in \mathbb{R}, \tag{1.7}$$

for a fixed integer $n \geq 2$.

This condition acts as the uniform continuity of h with order n, and plays an important role for the existence of viscosity solutions [10, 11]. We notice that (1.7) is fulfilled for $h(x) = |x|^{\bar{n}}$, $\bar{n} \in [2, \mathbf{n}]$.

In Section 2 we show that the value function $v_L(x) = \inf_{c \in \mathcal{A}_L} J(c)$ is a unique viscosity solution of (1.6), where $\mathcal{A}_L = \{c = (c_t) \in \mathcal{A} : |c_t| \leq L \text{ for all } t \geq 0\}$. Section 3 is devoted to the study of u that $u(x) := \lim_{L \to \infty} v_L(x)$ is a viscosity solution of (1.5).

2. Viscosity solutions

We here study the properties of the value function $v_L(x)$ and show that $v_L(x)$ is a viscosity solution of the Bellman equation (1.6) for any fixed $L > 0$, and then v_L converges to a viscosity solution u of the Bellman equation (1.5).

Given a continuous and degenerate elliptic map $H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we recall by [4] the definition of viscosity solutions of

$$H(x, w, w', w'') = 0 \quad \text{in } \mathbb{R}. \tag{2.1}$$

Definition 2.1. $w \in C(\mathbb{R})$ is called a viscosity subsolution (resp., supersolution) of (2.1) if, whenever for $\varphi \in C^2(\mathbb{R})$, $w - \varphi$ attains its local maximum (resp., minimum) at $x \in \mathbb{R}$, then

$$H(x, w(x), \varphi'(x), \varphi''(x)) \leq 0, \tag{2.2}$$

$$\text{resp., } H(x, w(x), \varphi'(x), \varphi''(x)) \geq 0. \tag{2.3}$$

Also $w \in C(\mathbb{R})$ is called a viscosity solution of (2.1) if it is both viscosity sub- and super-solution of (2.1).

According to Crandall et al. [4] and Fleming and Soner [7] this definition is equivalent to the following: for any $x \in \mathbb{R}$,

$$
\begin{aligned}
H(x, w(x), p, q) \leq 0 & \quad \text{for } (p, q) \in J^{2,+} w(x), \\
H(x, w(x), p, q) \geq 0 & \quad \text{for } (p, q) \in J^{2,-} w(x),
\end{aligned}
\tag{2.4}
$$

where $J^{2,+}$ and $J^{2,-}$ are the second-order superjets and subjets defined by

$$
J^{2,+} w(x) = \left\{ (p, q) \in \mathbb{R}^2 : \limsup_{y \to x} \frac{w(y) - w(x) - p(y - x) - (1/2)q|y - x|^2}{|y - x|^2} \leq 0 \right\},
$$

$$
J^{2,-} w(x) = \left\{ (p, q) \in \mathbb{R}^2 : \liminf_{y \to x} \frac{w(y) - w(x) - p(y - x) - (1/2)q|y - x|^2}{|y - x|^2} \geq 0 \right\}.
\tag{2.5}
$$

In order to obtain the viscosity property of v_L, we assume that there exists $\beta_0 \in (0, \beta)$ satisfying

$$
-\beta_0 + \sigma^2 n(2n - 1) + 2n|A| < 0,
\tag{2.6}
$$

and we set $f_k(x) = \gamma + |x|^k$ for any $2 \leq k \leq 2n$ and a constant $\gamma \geq 1$ chosen later.

2.1. Properties of viscosity solutions

LEMMA 2.2. *Assume (2.6). Then there exist $\gamma \geq 1$ and $\eta > 0$, depending on L, k, such that*

$$
-\beta_0 f_k + \frac{1}{2}\sigma^2 x^2 f_k'' + A x f_k' + \max_{|r| \leq L} (r^2 + r f_k') + \eta f_k \leq 0.
\tag{2.7}
$$

Further

$$
E\left[\int_0^\tau e^{-\beta_0 s} \eta f_k(x_s)\, ds + e^{-\beta_0 \tau} f_k(x_\tau) \right] \leq f_k(x) \quad \text{for } 2 \leq k \leq 2n,
\tag{2.8}
$$

$$
E\left[\sup_t e^{-\beta_0 t} f_k(x_t) \right] < \infty \quad \text{for } 2 \leq k \leq n,
\tag{2.9}
$$

where τ is any stopping time and x_t is the response to $(c_t) \in \mathscr{A}_L$.

Proof. By (2.6), we choose $\eta \in (0, \beta_0)$ such that

$$
-\beta_0 + \frac{1}{2}\sigma^2 k(k - 1) + k|A| + \eta < 0,
\tag{2.10}
$$

and then $\gamma \geq 1$ such that

$$\left(-\beta_0 + \frac{1}{2}\sigma^2 k(k-1) + k|A| + \eta\right)|x|^k + Lk|x|^{k-1} + (L^2 + \eta\gamma - \beta_0\gamma) \leq 0. \tag{2.11}$$

Then (2.7) is immediate. By (2.7) and Ito's formula, we deduce (2.8). Moreover, by moment inequalities for martingales we get

$$E\left[\sup_t e^{-\beta_0 t} f_k(x_t)\right] \leq f_k(x) + E\left[\sup_t \left|\int_0^t e^{-\beta_0 s} f_k'(x_s)\sigma x_s dw_s\right|\right]$$

$$\leq f_k(x) + KE\left[\left(\int_0^\infty e^{-2\beta_0 s}\sigma^2 |x_s|^{2k} ds\right)^{1/2}\right], \tag{2.12}$$

for some constant $K > 0$. Therefore (2.9) follows from this relation together with (2.8). □

THEOREM 2.3. *Assume (1.3), (1.4), (1.7), and (2.6). Then*

$$v_L \quad \textit{fulfills } (1.3),(1.4),(1.7), \tag{2.13}$$

and the dynamic programming principle holds, that is,

$$v_L(x) = \inf_{c \in \mathscr{A}_L} E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt + e^{-\beta\tau}v_L(x_\tau)\right], \tag{2.14}$$

for any stopping time τ.

Proof. We supress L of v_L for simplicity. The convexity of v follows from the same line as [5, Chapter 4, Lemma 10.6]. Let x_t^0 be the unique solution of

$$dx_t^0 = Ax_t^0 dt + \sigma x_t^0 dw_t, \qquad x_0^0 = x. \tag{2.15}$$

Then, by (1.4) and (2.8)

$$v(x) \leq E\left[\int_0^\infty e^{-\beta t}h(x_t^0)dt\right] \leq CE\left[\int_0^\infty e^{-\beta_0 t}f_n(x_t^0)dt\right] \leq \frac{Cf_n(x)}{\eta}. \tag{2.16}$$

For the solution y_t of (1.2) with $y_0 = y$, it is clear that $x_t - y_t$ fulfills (2.15) with initial condition $x - y$. We note by (2.10) with $k = n$ and Ito's formula that

$$E\left[e^{-\beta_0 t}|x_t^0|^n\right] \leq |x|^n. \tag{2.17}$$

Thus by (1.7) and (2.8)

$$
\begin{aligned}
|v(x) - v(y)| &\le \sup_{c \in \mathcal{A}_L} E\left[\int_0^\infty e^{-\beta t} |h(x_t) - h(y_t)| \, dt \right] \\
&\le \sup_{c \in \mathcal{A}_L} E\left[\int_0^\infty e^{-\beta t} \{ C_\rho |x_t - y_t|^n + \rho(1 + |x_t|^n + |y_t|^n) \} dt \right] \\
&\le \sup_{c \in \mathcal{A}_L} \int_0^\infty e^{-\beta t} \{ C_\rho |x - y|^n e^{\beta_0 t} + \rho(h_n(x) + h_n(y)) e^{\beta_0 t} \} dt \\
&\le \frac{1}{\beta - \beta_0} \left[C_\rho |x - y|^n + 2\rho\gamma(1 + |x|^n + |y|^n) \right].
\end{aligned}
\tag{2.18}
$$

Therefore we get (2.13).

To prove (2.14), we denote by $v^r(x)$ the right-hand side of (2.14). By the formal Markov property

$$
E\left[\int_\tau^\infty e^{-\beta t} \{ h(x_t) + |c_t|^2 \} dt \mid \mathscr{F}_\tau \right] = E\left[\int_0^\infty e^{-\beta(t+\tau)} \{ h(x_{\tau+t}) + |c_{\tau+t}|^2 \} dt \mid \mathscr{F}_\tau \right] = e^{-\beta\tau} J_{\tilde{c}}(x_\tau),
\tag{2.19}
$$

with $\tilde{c}$ equal to c shifted by τ. Thus

$$
\begin{aligned}
J_c(x) &= E\left[\int_0^\tau + \int_\tau^\infty e^{-\beta t} \{ h(x_t) + |c_t|^2 \} dt \right] \\
&= E\left[\int_0^\tau e^{-\beta t} \{ h(x_t) + |c_t|^2 \} dt \right] + E\left[\int_\tau^\infty \frac{e^{-\beta t} \{ h(x_t) + |c_t|^2 \} dt}{\mathscr{F}_\tau} \right] \\
&\ge E\left[\int_0^\tau e^{-\beta t} \{ h(x_t) + |c_t|^2 \} dt + e^{-\beta\tau} v_L(x_\tau) \right].
\end{aligned}
\tag{2.20}
$$

It is known in [7, 12] that this formal argument can be verified, and we deduce $v_L(x) \ge v^r(x)$.

To prove the reverse inequality, let $\rho > 0$ be arbitrary. We set

$$
V_c(x) = E\left[\int_0^\infty e^{-\beta t} \{ h(x_t) + |c_t|^2 \} dt \right].
\tag{2.21}
$$

By the same calculation as (2.18), there exists $C_\rho > 0$ such that

$$
|V_c(x) - V_c(y)| \le C_\rho |x - y|^n + \rho(1 + |x|^n + |y|^n).
\tag{2.22}
$$

Take $0 < \delta < 1$ with $C_\rho \delta^n < \rho$. Then, we have for $|x - y| < \delta$,

$$
\begin{aligned}
|v(x) - v(y)| &\leq \sup_{c \in \mathscr{A}_L} |V_c(x) - V_c(y)| \\
&\leq \rho(2 + |x|^n + |y|^n) \\
&\leq \rho[2 + |x|^n + 2^n(1 + |x|^n)] \\
&= \rho[(2 + 2^n) + (1 + 2^n)|x|^n] \\
&\leq \Xi_\rho(x) := \rho(2^n + 2)(1 + |x|^n).
\end{aligned}
\tag{2.23}
$$

Let $\{S_i\}$ be a sequence of disjoint subsets of $\mathbf{R}$ such that

$$
\operatorname{diam}(S_i) < \delta, \qquad \cup_i S_i = \mathbb{R}.
\tag{2.24}
$$

For any i, we take $x^{(i)} \in S_i$ and $c^{(i)} \in \mathscr{A}_L$ such that

$$
V_{c^{(i)}}(x^{(i)}) \leq \inf_{c \in \mathscr{A}_L} V_c(x^{(i)}) + \rho.
\tag{2.25}
$$

Define $c^\tau \in \mathscr{A}_L$ by

$$
c_t^\tau = c_t \mathbf{1}_{\{t < \tau\}} + c_{t-\tau}^{(i)} \mathbf{1}_{\{x_\tau \in S_i\}} \mathbf{1}_{\{t \geq \tau\}}, \quad \text{for } x_\tau \in S_i.
\tag{2.26}
$$

Hence,

$$
\begin{aligned}
V_{c^{(i)}}(x_\tau) &= V_{c^{(i)}}(x_\tau) - V_{c^{(i)}}(x^{(i)}) + V_{c^{(i)}}(x^{(i)}) \\
&\leq \Xi_\rho(x_\tau) + V_{c^{(i)}}(x^{(i)}) \\
&\leq \Xi_\rho(x_\tau) + \inf_{c \in \mathscr{A}_L} V_c(x^{(i)}) + \rho \\
&= \Xi_\rho(x_\tau) + v(x^{(i)}) + \rho \\
&\leq 2\Xi_\rho(x_\tau) + v(x_\tau) + \rho.
\end{aligned}
\tag{2.27}
$$

Now, by the definition of $v^r(x)$, we can find $c \in \mathscr{A}_L$ such that

$$
v^r(x) + \rho \geq E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\} dt + e^{-\beta \tau} v(x_\tau)\right].
\tag{2.28}
$$

Thus, using the formal Markov property [7], we have

$$
v^r(x) + \rho \geq \sum_i E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt + e^{-\beta\tau}(V_{c^{(i)}}(x_\tau) - 2\Xi_\rho(x_\tau) - \rho) : x_\tau \in S_i\right]
$$

$$
= E\left[\int_0^\tau e^{-\beta t}\{h(x_t^\tau) + |c_t^\tau|^2\}dt + \int_\tau^\infty e^{-\beta t}\{h(x_t^\tau) + |c_t^\tau|^2\}dt \mid \mathcal{F}_\tau\right]
$$

$$
- 2E[e^{-\beta\tau}\Xi_\rho(x_\tau)] - \rho \geq v(x) - 2\Xi_\rho(x) - \rho,
$$

(2.29)

where x_t^τ is the response to c_t^τ with $x_0^\tau = x_\tau$. Letting $\rho \to 0$, we deduce $v^r(x) \geq v(x)$, which completes the proof. □

LEMMA 2.4. *Let $h > 0$, then*

$$
\sup_{c \in \mathcal{A}_L} \sup_{0 \leq t \leq h} E|x_t|^2 < \infty,
$$

(2.30)

$$
\lim_{h \to 0} \sup_{c \in \mathcal{A}_L} E\left[\sup_{0 \leq t \leq h} |x_t - x|^2\right] = 0.
$$

(2.31)

Proof. By Ito's formula, we have

$$
|x_t|^2 = |x|^2 + \int_0^t \{2Ax_t|x_t|\operatorname{sgn}(x_t) + 2|x_t|\operatorname{sgn}(x_t)c_t + \sigma^2 x_t^2\}dt
$$

(2.32)

$$
+ \int_0^t 2|x_t|\operatorname{sgn}(x_t)dw_t.
$$

Let

$$
\tau_n = \begin{cases} \inf\{t : |x_t| > n\}, & \text{if } \{t : |x_t| > n\} \neq \varnothing, \\ \infty, & \text{if } \{t : |x_t| > n\} = \varnothing. \end{cases}
$$

(2.33)

Then

$$
E\left[|x_{t \wedge \tau_n}|^2\right] = |x|^2 + E\left[\int_0^{t \wedge \tau_n} \{2A + \sigma^2\}|x_t|^2 dt\right] + E\left[\int_0^{t \wedge \tau_n} 2|x_t|\operatorname{sgn}(x_t)c_t dt\right]
$$

$$
+ E\left[\int_0^{t \wedge \tau_n} \sigma x_t 2|x_t|\operatorname{sgn}(x_t)dw_t\right]
$$

$$
\leq |x|^2 + (2A + \sigma^2)E\left[\int_0^{t \wedge \tau_n} |x_t|^2 dt\right] + 2LE\left[\int_0^{t \wedge \tau_n}(1 + |x_t|^2)dt\right]
$$

$$
\leq (|x|^2 + 2Lh) + (2A + \sigma^2 + 2L)E\left[\int_0^{t \wedge \tau_n} |x_t|^2 dt\right]
$$

$$
\leq C_2 + C_1 \int_0^t E\left[|x_{s \wedge \tau_n}|^2\right]ds,
$$

(2.34)

for $C_1 = (2A+\sigma^2+2L), C_2 = (|x|^2+2Lh) > 0$. Set $E[|x_{t\wedge\tau_n}|^2] = \xi(t)$, then $\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2$. By Gronwall's lemma, we have $\xi(t) \leq C_2(1 + C_1 t e^{C_1 t}) = C_3 e^{C_4 t}$, for $C_3, C_4 > 0$. Hence $E[|x_{t\wedge\tau_n}|^2] \leq C_3 e^{C_4 t}$.

By Fatou's lemma, we obtain

$$E\left[|x_t|^2\right] \leq C_3 e^{C_4 t}. \tag{2.35}$$

Therefore, we obtain (2.30).

To prove (2.31), we have by (2.30)

$$E\left[\int_0^h |x_t|^2 dt\right] = \int_0^h E|x_t|^2 dt \leq h C_3 e^{C_4 h}, \tag{2.36}$$

and by the moment inequalities for local martingales [9] and Hölder's inequality

$$E\left[\left\{\left(\sup_{0\leq t\leq h}\left|\int_0^t \sigma x_t dw_t\right|\right)^2\right\}\right] \leq CE\left[\left(\int_0^h |x_t|^2 dt\right)\right] \leq C\left[\int_0^h E|x_t|^2 dt\right], \tag{2.37}$$

for some constant $C > 0$. Then

$$E\left[\sup_{0\leq t\leq h}|x_t - x|^2\right] \leq 3^2 \left(E\left[\left(\int_0^h |Ax_t|\,dt\right)^2 + \left(\int_0^h |c_t|\,dt\right)^2 + \left(\sup_{0\leq s\leq h}\left|\int_0^s \sigma x_t dw_t\right|\right)^2\right]\right)$$

$$\leq 3^2 \left(|A|^2 E\left[\left(\int_0^h |x_t|^2 dt\right)\left(\int_0^h 1^{2/(2-1)}dt\right)^{2-1}\right] + h^2 L^2 \right.$$

$$+ E\left[\left\{\left(\sup_{0\leq t\leq h}\left|\int_0^t \sigma x_t dw_t\right|\right)^2\right\}^{2/2}\right]\right)$$

$$\leq 3^2 \left(|A|^2 h^{2-1} E\left[\int_0^h |x_t|^2 dt\right] + h^2 L^2 + C\int_0^h E|x_t|^2 dt\right). \tag{2.38}$$

Hence, we have (2.31). □

THEOREM 2.5. *Assume (1.3), (1.4), (1.7), and (2.6). Then v_L is a viscosity solution of (1.6).*

Proof. Let $\varphi \in C^2(\mathbb{R})$ and let z be the maximizer of $v_L - \varphi$ with $v_L(z) = \varphi(z)$. Then (2.14) with the constant control $c_t = c \in \mathcal{A}_L$ gives

$$\varphi(z) \leq E\left[\int_0^s e^{-\beta t}\{h(x_t) + |c|^2\}dt + e^{-\beta s}v(x_s)\right]$$

$$\leq E\left[\int_0^s e^{-\beta t}\{h(x_t) + |c|^2\}dt + e^{-\beta s}\varphi(x_s)\right], \tag{2.39}$$

where

$$dx_t = [Ax_t + c]dt + \sigma x_t dw_t, \qquad x_0 = z. \tag{2.40}$$

We substract $\varphi(z)$ from both sides and apply Ito's formula to obtain

$$0 \le E\left[\int_0^s e^{-\beta t} \{h(x_t) + |c|^2\} dt + e^{-\beta s} \varphi(x_s) - \varphi(z) \right]$$

$$= E\left[\int_0^s e^{-\beta t} \{h(x_t) + |c|^2\} dt + \int_0^s e^{-\beta t} \left\{ -\beta\varphi + Ax\varphi' + \varphi'c + \frac{1}{2}\sigma^2 x^2 \varphi'' \right\}|_{x=x_t} dt \right]. \tag{2.41}$$

Divide by s and let $s \to 0$ and by dominated convergence theorem,

$$0 \le h(z) - \beta\varphi(z) + Az\varphi'(z) + \varphi'(z)c + |c|^2 + \frac{1}{2}\sigma^2 z^2 \varphi''(z). \tag{2.42}$$

Thus (2.3) is verified.

Let $\mathbb{O} = \mathbb{O}_x$ be a bounded neighborhood of x and let $\theta = \theta_x$ be the exit time of x_t from $\mathbb{O}_x$. Then, by (2.31)

$$P(\theta \le h) \le P\left(\sup_{0 \le s \le h} |x_s - x| \ge d(x, \partial\mathbb{O}) \right) \le \left(E\left[\sup_{0 \le s \le h} |x_s - x|^2 \right] \right) (d(x, \partial\mathbb{O}))^{-2}, \tag{2.43}$$

from which

$$\frac{1}{h} E[h - h \wedge \theta] \le \frac{1}{h} E[h - \theta : \theta \le h] \le P(\theta \le h) \longrightarrow 0 \quad \text{as } h \longrightarrow 0. \tag{2.44}$$

Let $\varphi \in C^2(\mathbb{R})$ and let z be the minimizer of $v_L - \varphi$ with $v_L(z) = \varphi(z)$. Take $\tau = h \wedge \theta_z$ in (2.14). Then by (2.14), there exists c_t^h with $|c_t^h| \le L$ such that

$$v_L(z) + h^2 \ge E\left[\int_0^\tau e^{-\beta t} \{h(x_t^h) + |c_t^h|^2\} dt + e^{-\beta\tau} v_L(x_\tau^h) \right], \tag{2.45}$$

where

$$dx_t^h = [Ax_t^h + c_t^h]dt + \sigma x_t^h dw_t, \qquad x_0^h = z. \tag{2.46}$$

Hence

$$\varphi(z) \ge E\left[\int_0^\tau e^{-\beta t} \{h(x_t^h) + |c_t^h|^2\} dt + e^{-\beta\tau} \varphi(x_\tau^h) \right] - h^2. \tag{2.47}$$

We substract $\varphi(z)$ from both sides and apply Ito's formula to obtain

$$0 \ge \frac{1}{h} E\left[\int_0^\tau e^{-\beta t} \{h(x_t^h) + |c_t^h|^2\} dt \right.$$

$$\left. + \int_0^\tau e^{-\beta t} \left\{ -\beta\varphi(x) + Ax\varphi'(x) + \varphi'(x)c_t^h + \frac{1}{2}\sigma^2 x^2 \varphi''(x) \right\}|_{x=x_t^h} dt \right] - h. \tag{2.48}$$

For any $\rho > 0$, by (1.7), (2.30), and (2.31), we have

$$
\sup_{0 \le t \le h} E[\,|h(x_t^h) - h(z)|\,] \le \sup_{0 \le t \le h} E\Big[\rho\big(1 + |x_t^h|^n + |z|^n\big) + C_\rho\,|x_t^h - z|^n\Big]
$$
$$
\longrightarrow 0 \quad \text{as } h \longrightarrow 0, \ \rho \longrightarrow 0.
$$
(2.49)

Hence $\lim_{h \to 0} (1/h) E[\int_0^h e^{-\beta t} h(x_t^h) dt] = h(z)$. Set

$$
\Big\{h(x_t^h) + |c_t^h|^2\Big\} + \Big\{-\beta\varphi(x) + Ax\varphi'(x) + \varphi'(x)c_t^h + \frac{1}{2}\sigma^2 x^2 \varphi''(x)\Big\}\Big|_{x=x_t^h} = h(x_t).
$$
(2.50)

By (1.4), (2.30), (2.44), and Schwarz's inequality, we obtain

$$
\frac{1}{h}\left|E\Big[\int_0^h h(x_t)\,dt\Big] - E\Big[\int_0^{h \wedge \theta} h(x_t)\,dt\Big]\right|
$$
$$
= \frac{1}{h}E\Big[\int_{h \wedge \theta}^h |h(x_t)|\,dt\Big]
$$
$$
\le \frac{1}{h}E\Big[\int_{h \wedge \theta}^h K(1 + |x_t|^n)\,dt\Big]
$$
$$
\le \frac{K}{h}E\Big[\Big(\int_{h \wedge \theta}^h 1\,dt\Big)^{1/2} \Big(\int_0^h (1 + |x_t|^n)^2\,dt\Big)^{1/2}\Big]
$$
(2.51)
$$
\le \frac{K}{h}\Big(E\Big[\int_{h \wedge \theta}^h 1\,dt\Big]\Big)^{1/2} \Big(E\Big[\int_0^h (1 + |x_t|^n)^2\,dt\Big]\Big)^{1/2}
$$
$$
= K\Big(\frac{1}{h}E[h - h \wedge \theta]\Big)^{1/2} \Big(\frac{1}{h}E\Big[\int_0^h (1 + |x_t|^n)^2\,dt\Big]\Big)^{1/2}
$$
$$
\longrightarrow 0 \quad \text{as } h \longrightarrow 0.
$$

Thus we have

$$
0 \ge \liminf_{h \to \infty} \frac{1}{h}E\Big[\int_0^h e^{-\beta t}\Big\{h(x_t^h) + |c_t^h|^2\Big\}dt
$$
$$
+ \int_0^h e^{-\beta t}\Big\{-\beta\varphi(x) + Ax\varphi'(x) + \varphi'(x)c_t^h + \frac{1}{2}\sigma^2 x^2 \varphi''(x)\Big\}\Big|_{x=x_t^h} dt\Big]
$$
$$
\ge \liminf_{h \to \infty} \frac{1}{h}E\Big[\int_0^h e^{-\beta t}\Big\{h(z) + |c_t^h|^2\Big\}dt
$$
$$
+ \int_0^h e^{-\beta t}\Big\{-\beta\varphi(x) + Ax\varphi'(x) + \varphi'(x)c_t^h + \frac{1}{2}\sigma^2 x^2 \varphi''(x)\Big\}\Big|_{x=z} dt\Big]
$$
$$
\ge h(z) - \beta\varphi(z) + Az\varphi'(z) + \min_{|c| \le L}\big(|c|^2 + \varphi'(z)c\big) + \frac{1}{2}\sigma^2 z^2 \varphi''(z),
$$
(2.52)

which implies (2.2) and completes the proof. $\qquad\square$

THEOREM 2.6 (Apostol [1] (Dini)). *Suppose that X is a compact metric space and $\{v_L(x)\}$ is a sequence of continuous functions that converges pointwise to a continuous function u on X. Suppose also that the sequence is pointwise nonincreasing, that is, for each $x \in X$, there exists $v_L(x) \geq v_{L+1}(x)$, for all $L \in \mathbb{N}$. Then $\{v_L(x)\}$ converges uniformly to u on X.*

2.2. Uniqueness of HJB.

The most important feature of the theory of viscosity solution is the powerful uniqueness theorem. In the context of optimal control problems, value function is the unique viscosity solutions.

In this section we give a detailed proof of uniqueness result for the quadratic control problem that v is a unique viscosity solution of (1.6). The references for these uniqueness results are Bardi and Capuzzo-Dolcetta [2] and Flacone and Makridakis [6].

THEOREM 2.7. *Assume (1.3), (1.4), (1.7), and (2.6). Let v_i (i = 1, 2) be two viscosity solutions of (1.6) satisfying (2.13). Then $v_1 = v_2$.*

Proof. We first note (2.6) and there exists $n < k < n + 1$ such that

$$-\beta \psi_k + \frac{1}{2}\sigma^2 x^2 \psi_k'' + Ax\psi_k' + \min_{r \in \mathbb{R}} \left(|r|^2 + r\psi_k' \right) \leq 0, \tag{2.53}$$

where $\psi_k(x) = (1 + |x|^k)$. Indeed, by (2.6) we choose $\vartheta \in (0, \beta)$ such that

$$-\beta + \frac{1}{2}k(k-1)\sigma^2 + k|A| + \vartheta < 0. \tag{2.54}$$

By (2.54), we have

$$\left(-\beta + \frac{1}{2}k(k-1)\sigma^2 + k|A| + \vartheta \right) |x|^k - \frac{k^2}{4} \left(|x|^{k-1} \right)^2 - \beta - \vartheta |x|^k < 0. \tag{2.55}$$

Then (2.53) is immediate.

Suppose that $v_1(x_0) - v_2(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Then we find $\eta > 0$ such that

$$\sup_{x \in \mathbb{R}} \left[v_1(x) - v_2(x) - 2\eta \psi_k(x) \right] > 0. \tag{2.56}$$

Since

$$v_1(x) - v_2(x) - 2\eta \psi_k(x) \leq \overline{K} \left(1 + |x|^n \right) - 2\eta \left(1 + |x|^k \right) \longrightarrow -\infty \quad \text{as } x \longrightarrow \infty, \tag{2.57}$$

there exists $\overline{x} \in \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} \left[v_1(x) - v_2(x) - 2\eta \psi_k(x) \right] = v_1(\overline{x}) - v_2(\overline{x}) - 2\eta \psi_k(\overline{x}) > 0. \tag{2.58}$$

Define

$$\Phi(x, y) = v_1(x) - v_2(y) - \frac{m}{2}|x - y|^2 - \eta(\psi_k(x) + \psi_k(y)), \tag{2.59}$$

for any $m > 0$. It is clear that

$$\Phi(x, y) \leq C(1 + |x|^n + |y|^n) - \eta(2 + |x|^k + |y|^k) \longrightarrow -\infty \quad \text{as } x, y \longrightarrow \infty, \tag{2.60}$$

where $C > \max\{\overline{K}, \rho\}$. Hence we find $(x_m, y_m) \in \mathbb{R}^2$ such that

$$\Phi(x_m, y_m) = \sup_{x, y} \Phi(x, y)$$

$$= v_1(x_m) - v_2(y_m) - \frac{m}{2}|x_m - y_m|^2 - \eta(\psi_k(x_m) + \psi_k(y_m)) \tag{2.61}$$

$$\geq v_1(\overline{x}) - v_2(\overline{x}) - 2\eta\psi_k(\overline{x}) > 0,$$

from which

$$\frac{m}{2}|x_m - y_m|^2 < v_1(x_m) - v_2(y_m) - \eta(\psi_k(x_m) + \psi_k(y_m))$$

$$\leq C(2 + |x_m|^n + |y_m|^n) - \eta(2 + |x_m|^k + |y_m|^k) \tag{2.62}$$

$$\longrightarrow -\infty \quad \text{as } |x_m|, |y_m| \longrightarrow \infty,$$

where $C > \max\{\overline{K}, \rho\}$. Thus we deduce that the sequences $\{x_m\}, \{y_m\}$ are bounded and then $\{m|x_m - y_m|^2\}$ is bounded by some constant $C > 0$, and

$$|x_m - y_m| \leq \left(\frac{C}{m}\right)^{1/2} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{2.63}$$

Now, we claim that

$$m|x_m - y_m|^2 \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{2.64}$$

$$x_m, y_m \longrightarrow \tilde{x} \quad \text{or} \quad \tilde{y} \text{ as } m \longrightarrow \infty. \tag{2.65}$$

Indeed, by the definition of (x_m, y_m),

$$\Phi(x_m, y_m) \geq v_1(x_m) - v_2(x_m) - 2\eta\psi_k(x_m). \tag{2.66}$$

Hence, by (1.7) and (2.61)

$$\frac{m}{2}|x_m - y_m|^2 \leq v_2(x_m) - v_2(y_m) + \eta(\psi_k(x_m) - \psi_k(y_m))$$

$$= v_2(x_m) - v_2(y_m) + \eta(|x_m|^k - |y_m|^k)$$

$$\leq v_2(x_m) - v_2(y_m) + \eta(n+1)C^{k-1}|x_m - y_m|$$

$$\leq C_\rho|x_m - y_m|^n + \rho(1 + |x_m|^n + |y_m|^n) + \eta(n+1)C^{k-1}|x_m - y_m|. \tag{2.67}$$

Letting $m \to \infty$ and then $\rho \to 0$, we obtain (2.64). Moreover, by (2.63) we have (2.65) taking a subsequence if necessary. Equation (2.63) implies $\tilde{x} = \tilde{y}$. Passing to the limit in (2.61), we get

$$v_1(\tilde{x}) - v_2(\tilde{x}) - 2\eta(1 + |\tilde{x}|^k) > 0. \tag{2.68}$$

$\square$

We apply Ishii's lemma below to

$$
\begin{aligned}
V_1(x) &= v_1(x) - \eta\psi_k(x), \\
V_2(y) &= v_2(y) + \eta\psi_k(y).
\end{aligned}
\tag{2.69}
$$

LEMMA 2.8 (Ishii). *Let V_1, $-V_2$ be upper semicontinuous in an open domain, and set*

$$\Phi(x,y) = V_1(x) - V_2(y) - \frac{m}{2}|x - y|^2. \tag{2.70}$$

Let $(\hat{x}, \hat{y})$ be the local maximizer of $\Phi(x, y)$. Then there exist symmetric matrices X_1, X_2 such that

$$
\begin{aligned}
(m(\hat{x} - \hat{y}), X_1) &\in \overline{J}^{2,+} V_1(\hat{x}), \\
(m(\hat{x} - \hat{y}), X_2) &\in \overline{J}^{2,-} V_2(\hat{y}), \\
\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} &\le 3m \begin{pmatrix} I & -I \\ I & I \end{pmatrix}, \quad I = identity,
\end{aligned}
\tag{2.71}
$$

where

$$
\begin{aligned}
\overline{J}^{2,\pm} V_1(x) = \{ (p, X_1) : \exists x_r \longrightarrow x, \exists (p_r, X_r) \in J^{2,\pm} V_1(x_r), \\
(V_1(x_r), p_r, X_r) \longrightarrow (V_1(x), p, X_1) \}.
\end{aligned}
\tag{2.72}
$$

Proof. For the proof, see Crandall et al. [4, Theorem 3.2], Fleming and Soner [7, Lemma 6.1, page 238], and Ishii [8, Lemma 1, page 149].

We remark that if $V_1, V_2 \in C^2$, then

$$\Phi_x(\hat{x}, \hat{y}) = \Phi_y(\hat{x}, \hat{y}) = 0, \tag{2.73}$$

from which

$$V_1'(\hat{x}) = m(\hat{x} - \hat{y}), \qquad V_2'(\hat{y}) = -m(\hat{x} - \hat{y}). \tag{2.74}$$

Since

$$\Phi_{xx} = V_1''(x) - m, \qquad \Phi_{xy} = m, \qquad \Phi_{yy} = -V_2''(y) - m, \tag{2.75}$$

the maximum principle gives

$$0 \geq D^2\Phi(\hat{x},\hat{y}) = \begin{pmatrix} V_1''(\hat{x}) & 0 \\ 0 & -V_2''(\hat{y}) \end{pmatrix} - m \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.76}$$

We obtain $X_1, X_2 \in \mathbb{R}^1$ such that

$$(m(x_m - y_m), X_1) \in \overline{J}^{2,+} V_1(x_m),$$

$$(m(x_m - y_m), X_2) \in \overline{J}^{2,-} V_2(y_m),$$

$$-3m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq 3m \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad I = \text{identity}, \tag{2.77}$$

where

$$\overline{J}^{2,\pm} V_i(x) = \{(p,X) : \exists x_r \longrightarrow x, \exists (p_r, X_r) \in J^{2,\pm} V_i(x_r),$$

$$(V_i(x_r), p_r, X_r) \longrightarrow (V_i(x), p, X)\}, \quad i = 1, 2. \tag{2.78}$$

Recall that

$$J^{2,+} v_1(x) = \{(p + \eta k|x|^{k-1} \operatorname{sgn}(x), X + \eta k(k-1)|x|^{k-2}) : (p,X) \in J^{2,+} V_1(x)\},$$

$$J^{2,-} v_2(y) = \{(p - \eta k|y|^{k-1} \operatorname{sgn}(y), X - \eta k(k-1)|y|^{k-2}) : (p,X) \in J^{2,-} V_2(y)\}. \tag{2.79}$$

Hence

$$(p_1, \overline{X}_1) := \left(m(x_m - y_m) + \eta k|x_m|^{k-1} \operatorname{sgn}(x_m), X_1 + \eta k(k-1)|x_m|^{k-2}\right)$$

$$\in \overline{J}^{2,+} v_1(x_m),$$

$$(p_2, \overline{X}_2) := \left(m(x_m - y_m) - \eta k|x_m|^{k-1} \operatorname{sgn}(y_m), X_2 - \eta k(k-1)|y_m|^{k-2}\right) \tag{2.80}$$

$$\in \overline{J}^{2,-} v_2(y_m),$$

$$x_m^2 X_1 \leq y_m^2 X_2.$$

By virtue of (1.6), (2.2) and (2.3) give

$$-\beta v_1(x) + \frac{1}{2}\sigma^2 x^2 \overline{X}_1 + Ax p_1 - \frac{|p_1|^2}{4} + h(x)|_{x=x_m} \geq 0,$$

$$-\beta v_2(y) + \frac{1}{2}\sigma^2 y^2 \overline{X}_2 + Ay p_2 - \frac{|p_2|^2}{4} + h(y)|_{y=y_m} \leq 0. \tag{2.81}$$

Putting these inequalities together, we get

$$\beta[v_1(x_m) - v_2(y_m)]$$

$$\leq \frac{1}{2}\sigma^2\left(x_m{}^2\overline{X}_1 - y_m{}^2\overline{X}_2\right) + A\left(x_m p_1 - y_m p_2\right)$$

$$- \frac{1}{4}\left(\left(|p_1|\right)^2 - \left(|p_2|\right)^2\right) + h(x_m) - h(y_m)$$

$$\leq \frac{1}{2}\sigma^2\eta k(k-1)\left[|x_m|^k + |y_m|^k\right] + Am(x_m - y_m)^2 \qquad (2.82)$$

$$+ A\eta k\left[|x_m|^k + |y_m|^k\right]$$

$$- \frac{1}{4}\left[\left(|m(x_m - y_m) + \eta k|x_m|^{k-1}\,\text{sgn}\,(x_m)|\right)^2\right.$$

$$\left. - \left(|m(x_m - y_m) - \eta k|y_m|^{k-1}\,\text{sgn}\,(y_m)|\right)^2\right] + h(x_m) - h(y_m).$$

Letting $m \to \infty$, we have

$$\beta[v_1(\tilde{x}) - v_2(\tilde{x})]$$

$$\leq 2\eta\left[\frac{1}{2}\sigma^2\tilde{x}^2 k(k-1)|\tilde{x}|^{k-2} + A\tilde{x}k|\tilde{x}|^{k-1} - \frac{k^2}{4}\left(|\tilde{x}|^{k-1}\right)^2\right] \leq 2\eta\beta\left(1 + |\tilde{x}|^k\right),$$

$$(2.83)$$

which follows from (2.53). This is contrary of (2.68), completing the proof. □

3. Stability of viscosity solutions of u

THEOREM 3.1. *Assume (1.3), (1.4), (1.7), and (2.6). Then $u(x)$ is a viscosity solution of (1.5), which satisfies (1.4) and (1.7).*

Proof. Combining Theorems 2.5 and 2.6, we get the assertion by the stability result of Fleming and Soner [7, Lemma 6.2, page 73] as follows.

Let $\varphi \in C^2(\mathbb{R})$ and let z be the maximizer of $u - \varphi$ such that $u(z) - \varphi(z) > u(x) - \varphi(x)$ in the closed ball $\overline{B}(z,\delta)$ with radius δ of $z \neq x$.

By Theorem 2.6, $v_L - \varphi$ attains a local maximum at some $z_L \in \overline{B}(z,\delta)$. Take a subsequence $\{z_{L'}\}$ of $\{z_L\}$ such that

$$z_{L'} \longrightarrow z' \in \overline{B}(z,\delta). \qquad (3.1)$$

By Theorem 2.6

$$(v_{L'} - \varphi)(z_{L'}) \longrightarrow (u - \varphi)(z'). \qquad (3.2)$$

Since

$$(v_{L'} - \varphi)(z_{L'}) > (v_{L'} - \varphi)(x), \quad x \in \overline{B}(z,\delta), \qquad (3.3)$$

we have

$$(u - \varphi)(z') \geq (u - \varphi)(x), \quad x \in \overline{B}(z, \delta). \tag{3.4}$$

In particular,

$$(u - \varphi)(z') \geq (u - \varphi)(z). \tag{3.5}$$

Thus we deduce

$$z = z', \tag{3.6}$$

and the convergence of the whole sequence.

Now it follows from Theorem 2.5 that

$$-\beta v_L + \frac{1}{2}\sigma^2 x^2 \varphi'' + Ax\varphi' + \min_{|c| \leq L} \left(|c|^2 + c\varphi' \right) + h(x)|_{x=z_L} \geq 0. \tag{3.7}$$

Note that

$$\min_{|c| \leq L} \left(|c|^2 + c\zeta \right) \longrightarrow \min_c \left(|c|^2 + c\zeta \right) \text{ locally uniformly in } \mathcal{R} \quad \text{as } L \longrightarrow \infty. \tag{3.8}$$

Letting $L \to \infty$, we get

$$-\beta u + \frac{1}{2}\sigma^2 x^2 \varphi'' + Ax\varphi' + \min_c \left(|c|^2 + c\varphi' \right) + h(x)|_{x=z} \geq 0. \tag{3.9}$$

Again a similar argument gives the proof for the minimizer of $u - \varphi$, that is,

$$-\beta u + \frac{1}{2}\sigma^2 x^2 \varphi'' + Ax\varphi' + \min_c \left(|c|^2 + c\varphi' \right) + h(x)|_{x=z} \leq 0. \tag{3.10}$$

Therefore, we see that u is a viscosity solution of (1.5) and then by Theorem 2.3, it is clear that u fulfills (1.4) and (1.7). □

Stochastic control problem: in general we can further study a stochastic control problem for linear degenerate systems to minimize the discounted expected cost:

$$J(c) = E\left[\int_0^\infty e^{-\alpha t} \{h(x_t) + |c_t|^m\} dt \right], \tag{3.11}$$

over $c \in \mathcal{A}$ subject to the degenerate stochastic differential equation (1.2) and a continuous function h on $\mathbb{R}$ such that (1.4) and (1.7), in addition

$$k_0 |x|^m - k_1 \leq h(x), \tag{3.12}$$

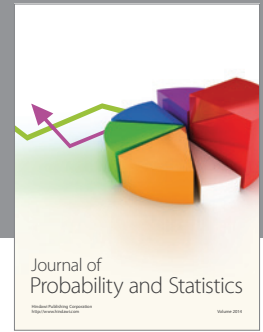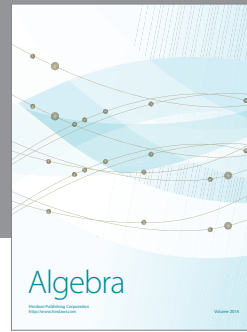for some constants $k_0, k_1 > 0$ and for a fixed integer $m \geq 2$.

Acknowledgment

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