(γ, β)-PS-IRRESOLUTE AND
(γ, β)-PS-CONTINUOUS FUNCTIONS

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Abstract: This paper introduces some new types of functions called (γ, β)-PS-irresolute and (γ, β)-PS-continuous by using γ-PS-open sets in topological spaces (X, τ). From γ-PS-open and γ-PS-closed sets, some other types of γ-PS-functions can also be defined. Moreover, some basic properties and preservation theorems of these functions are obtained. In addition, we investigate basic characterizations and properties of these γ-PS-functions. Finally, some compositions of these γ-PS-functions are given.

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1. Introduction

Kasahara [7] defined the concept of α-closed graphs of an operation on τ. Later, Ogata [10] renamed the operation α as γ operation on τ. He defined γ-open sets and introduced the notion of τγ which is the class of all γ-open sets in a topological space (X, τ). Further study by Krishnan and Balachandran ([8], [9]) defined two types of sets called γ-preopen and γ-semiopen sets. Recently, Asaad, Ahmad and Omar [1] introduced the notion of γ-regular-open sets. They
also introduced the notion of $\gamma$-$P_S$-open sets [2] which lies strictly between the classes of $\gamma$-regular-open set and $\gamma$-preopen set. By using this set, they defined a new type of function called $\gamma$-$P_S$-continuous and studies some of its basic properties [3].

In the present paper, we define some new types of $\gamma$-$P_S$- functions called $(\gamma, \beta)$-$P_S$-irresolute and $(\gamma, \beta)$-$P_S$-continuous by using $\gamma$-$P_S$-open sets in topological spaces $(X, \tau)$. In addition, we give some basic characterizations and properties of these $\gamma$-$P_S$- functions by using $\gamma$-$P_S$-open and $\gamma$-$P_S$-closed sets are introduced. Finally, some compositions of these $\gamma$-$P_S$- functions are given.

2. Preliminaries

Throughout this paper, spaces $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) will always mean topological spaces on which no separation axioms are assumed unless explicitly stated. An operation $\gamma$ on the topology $\tau$ on $X$ is a mapping $\gamma: \tau \rightarrow P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of $X$ and $\gamma(U)$ denotes the value of $\gamma$ at $U$ [10]. A nonempty subset $A$ of a space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-open if for each $x \in A$, there exists an open set $U$ such that $x \subseteq U$ and $\gamma(U) \subseteq A$ [10]. The complement of a $\gamma$-open set is called a $\gamma$-closed. The $\tau_\gamma$-closure of a subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is defined as the intersection of all $\gamma$-closed sets of $X$ containing $A$ and it is denoted by $\tau_\gamma Cl(A)$ [10], and the $\tau_\gamma$-interior of a subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is defined as the union of all $\gamma$-open sets of $X$ contained in $A$ and it is denoted by $\tau_\gamma Int(A)$ [9]. A topological space $(X, \tau)$ is said to be $\gamma$-regular if for each $x \in X$ and for each open neighborhood $V$ of $x$, there exists an open neighborhood $U$ of $x$ such that $\gamma(U) \subseteq V$ [10]. Throughout of this paper, $\gamma$ and $\beta$ be operations on $\tau$ and $\sigma$ respectively.

Now we begin to recall some known notions which are useful in the sequel.

Definition 2.1. A subset $A$ of a topological space $(X, \tau)$ is said to be:

1. $\gamma$-regular-open if $A = \tau_\gamma Int(\tau_\gamma Cl(A))$ and $\gamma$-regular-closed if $A = \tau_\gamma Cl(\tau_\gamma Int(A))$ [1].

2. $\gamma$-preopen if $A \subseteq \tau_\gamma Int(\tau_\gamma Cl(A))$ and $\gamma$-preclosed if $\tau_\gamma Cl(\tau_\gamma Int(A)) \subseteq A$ [8].

3. $\gamma$-semiopen if $A \subseteq \tau_\gamma Cl(\tau_\gamma Int(A))$ and $\gamma$-semiclosed if $\tau_\gamma Int(\tau_\gamma Cl(A)) \subseteq A$ [9].
4. \( \gamma \)-dense if \( \tau_\gamma Cl(A) = X \) [6].

**Definition 2.2.** [2] A \( \gamma \)-preopen subset \( A \) of a topological space \( (X, \tau) \) is called \( \gamma \)-\( P_S \)-open if for each \( x \in A \), there exists a \( \gamma \)-semiclosed set \( F \) such that \( x \in F \subseteq A \). The complement of a \( \gamma \)-\( P_S \)-open set of \( X \) is called \( \gamma \)-\( P_S \)-closed.

The class of all \( \gamma \)-\( P_S \)-open and \( \gamma \)-preopen subsets of a topological space \( (X, \tau) \) are denoted by \( \tau_\gamma P_S O(X) \) and \( \tau_\gamma PO(X) \) respectively.

**Lemma 2.3.** [2] A subset \( A \) of \( X \) is \( \gamma \)-\( P_S \)-open if and only if \( A \) is \( \gamma \)-preopen set and it is a union of \( \gamma \)-semiclosed sets.

**Definition 2.4.** A subset \( N \) of a topological space \( (X, \tau) \) is called a \( \gamma \)-\( P_S \)-neighbourhood of a point \( x \in X \), if there exists a \( \gamma \)-\( P_S \)-open set \( U \) in \( X \) containing \( x \) such that \( U \subseteq N \).

**Definition 2.5.** [2] For any subset \( A \) of a space \( X \). Then:

1. the \( \gamma \)-\( P_S \)-boundary of \( A \) is defined as \( \tau_\gamma P_S Cl(A) \setminus \tau_\gamma P_S Int(A) \) and it is denoted by \( \tau_\gamma P_S Bd(A) \).

2. the \( \gamma \)-\( P_S \)-derived set of \( A \) is defined as \( \{ x : \text{for every} \ \gamma \text{-}P_S \text{-open } U \in X \ \text{containing} \ x, \ U \cap A \setminus \{ x \} \neq \phi \} \) and it is denoted by \( \tau_\gamma P_S D(A) \).

**Lemma 2.6.** [2] For any subset \( A \) of a space \( X \). Then the following statements are true:

1. \( \tau_\gamma P_S Cl(A) \) is the smallest \( \gamma \)-\( P_S \)-closed set of \( X \) containing \( A \).

2. \( \tau_\gamma P_S Int(A) \) is the largest \( \gamma \)-\( P_S \)-open set of \( X \) contained in \( A \).

3. \( A \) is \( \gamma \)-\( P_S \)-closed if and only if \( \tau_\gamma P_S Cl(A) = A \), and \( A \) is \( \gamma \)-\( P_S \)-open if and only if \( \tau_\gamma P_S Int(A) = A \).

4. \( \tau_\gamma P_S Cl(A) = X \setminus \tau_\gamma P_S Int(X \setminus A) \) and \( \tau_\gamma P_S Int(A) = X \setminus \tau_\gamma P_S Cl(X \setminus A) \).

5. \( A \) is \( \gamma \)-\( P_S \)-closed if and only if \( \tau_\gamma P_S Bd(A) \subseteq A \).

6. \( \tau_\gamma P_S D(A) \subseteq \tau_\gamma P_S Cl(A) \).

7. \( A \) is \( \gamma \)-\( P_S \)-closed if and only if \( \tau_\gamma P_S D(A) \subseteq A \).

**Definition 2.7.** [4] A subset \( A \) of a space \( (X, \tau) \) is said to be \( \gamma \)-\( P_S \)-generalized closed \(( \gamma \)-\( P_S \)-\( g \)-closed\)) if \( \tau_\gamma P_S Cl(A) \subseteq G \) whenever \( A \subseteq G \) and \( G \) is a \( \gamma \)-\( P_S \)-open set in \( X \).

**Definition 2.8.** A topological space \( (X, \tau) \) is said to be:
1. $\gamma$-locally indiscrete if every $\gamma$-open subset of $X$ is $\gamma$-closed, or every $\gamma$-closed subset of $X$ is $\gamma$-open [1].

2. $\gamma$-hyperconnected if every nonempty $\gamma$-open subset of $X$ is $\gamma$-dense [1].

3. $\gamma$-$P_S$-$T_\frac{1}{2}$ if every $\gamma$-$P_S$-$g$-closed set of $X$ is $\gamma$-$P_S$-closed [4].

4. $\gamma$-semi-$T_1$ if for each pair of distinct points $x, y$ in $X$, there exist two $\gamma$-semiopen sets $U$ and $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$ [9].

**Theorem 2.9.** The following statements are true for any space $(X, \tau)$:

1. If $X$ is $\gamma$-locally indiscrete, then $\tau_{\gamma}$-$P_SO(X) = \tau_{\gamma}$ [2].

2. If $X$ is $\gamma$-semi$T_1$, then $\tau_{\gamma}$-$P_SO(X) = \tau_{\gamma}$-$PO(X)$ [2].

3. If $X$ is $\gamma$-hyperconnected if and only if $\tau_{\gamma}$-$P_SO(X) = \{\phi, X\}$ [2].

4. $X$ is $\gamma$-$P_S$-$T_\frac{1}{2}$ if and only if for each element $x \in X$, the set $\{x\}$ is $\gamma$-$P_S$-closed or $\gamma$-$P_S$-open [4].

5. $X$ is $\gamma$-regular, then $\tau_{\gamma} = \tau$ [10].

**Definition 2.10.** Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces. A function $f: (X, \tau) \to (Y, \sigma)$ is called:

1. $\gamma$-$P_S$-continuous if for each $P_S$-open set $V$ of $Y$ containing $f(x)$, there exists a $\gamma$-$P_S$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$ [3].

2. $(\gamma, \beta)$-precontinuous if for each $\beta$-preopen set $V$ of $Y$ containing $f(x)$, there exists a $\gamma$-preopen set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$ [8].

3. $\gamma$-continuous if for each open set $V$ of $Y$ containing $f(x)$, there exists a $\gamma$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$ [5].

4. $\beta$-$P_S$-open (resp., $\beta$-open and $\beta$-$P_S$-closed) if for every open (resp., open and closed) set $V$ of $X$, $f(V)$ is $\beta$-$P_S$-open (resp., $\beta$-open and $\beta$-$P_S$-closed) set in $Y$ [3].
3. \((\gamma, \beta)-P_S\)-Irresolute and \((\gamma, \beta)-P_S\)-Continuous Functions

In this section, we introduce three types of \(\gamma-P_S\)-functions called \((\gamma, \beta)-P_S\)-irresolute and \((\gamma, \beta)-P_S\)-continuous by using \(\gamma-P_S\)-open set. Also we give relations between these functions and \(\gamma-P_S\)-continuous function.

**Definition 3.1.** Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces. A function \(f: (X, \tau) \to (Y, \sigma)\) is called \((\gamma, \beta)-P_S\)-irresolute (resp., \((\gamma, \beta)-P_S\)-continuous) at a point \(x \in X\) if for each \(\beta-P_S\)-open (resp., \(\beta\)-open) set \(V\) of \(Y\) containing \(f(x)\), there exists a \(\gamma-P_S\)-open set \(U\) of \(X\) containing \(x\) such that \(f(U) \subseteq V\). If \(f\) is \((\gamma, \beta)-P_S\)-irresolute (resp., \((\gamma, \beta)-P_S\)-continuous) at every point \(x\) in \(X\), then \(f\) is said to be \((\gamma, \beta)-P_S\)-irresolute (resp., \((\gamma, \beta)-P_S\)-continuous).

**Remark 3.2.** It is clear from the Definition 2.10 (1) and Definition 3.1 that every \(\gamma-P_S\)-continuous function is \((\gamma, \beta)-P_S\)-continuous since every \(\beta\)-open set is open, where \(\beta\) is an operation on \(\sigma\). However, the converse is not true in general as it can be seen from the following example.

**Example 3.3.** Let \(X = \{a, b, c\}\) with the topology \(\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}\) and \(Y = \{1, 2, 3\}\) with the topology \(\sigma = \{\phi, Y, \{2\}, \{3\}, \{2, 3\}\}\). Define operations \(\gamma: \tau \to P(X)\) and \(\beta: \sigma \to P(Y)\) as follows: for every \(A \in \tau\) and \(B \in \sigma\)

\[
\gamma(A) = \begin{cases} 
A & \text{if } a \in A \\
\text{Cl}(A) & \text{if } a \notin A
\end{cases}
\]

\[
\beta(B) = \begin{cases} 
B & \text{if } B = \{2\} \\
\text{Cl}(B) & \text{if } B \neq \{2\}
\end{cases}
\]

Then \(\sigma_\beta = \{\phi, \{2\}, Y\}\).

Let \(f: (X, \tau) \to (Y, \sigma)\) be a function defined as follows:

\[
f(x) = \begin{cases} 
2 & \text{if } x = a \\
3 & \text{if } x = b \\
1 & \text{if } x = c
\end{cases}
\]

Clearly, \(\tau = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}\), \(\tau_\gamma \text{-} P_S O(X) = \{\phi, \{a\}, \{a, c\}, \{b, c\}, X\}\) and \(\sigma_\beta = \{\phi, \{2\}, Y\}\). Let \(f: (X, \tau) \to (X, \sigma)\) be a function defined as follows:

\[
f(x) = \begin{cases} 
2 & \text{if } x = a \\
3 & \text{if } x = b \\
1 & \text{if } x = c
\end{cases}
\]

Then \(f\) is \((\gamma, \beta)-P_S\)-continuous, but it is not \(\gamma-P_S\)-continuous since \(\{3\}\) is an open set in \((Y, \sigma)\) containing \(f(b) = 3\), but there exist no \(\gamma-P_S\)-open set \(U\) in \((X, \tau)\) containing \(b\) such that \(f(U) \subseteq \{3\}\).
**Remark 3.4.** The relation between \((\gamma, \beta)\)-\(P_S\)-irresolute function and \((\gamma, \beta)\)-\(P_S\)-continuous function are independent. Similarly the relation between \((\gamma, \beta)\)-\(P_S\)-irresolute function and \(\gamma\)-\(P_S\)-continuous function are independent, as shown from the following examples.

**Example 3.5.** Let \((X, \tau)\) be a topological space as in Example 3.3. Suppose that \(Y = \{1, 2, 3\}\) and \(\sigma = \{\phi, Y, \{2\}, \{2, 3\}\}\) be a topology on \(Y\). Define an operation \(\beta\) on \(\sigma\) such that \(\beta: \sigma \to P(Y)\) by \(\beta(B) = B\) for all \(B \in \sigma\). Then \(\sigma_{\beta-P_S}(Y) = \{\phi, Y\}\).

Let \(f: (X, \tau) \to (Y, \sigma)\) be a function defined as follows:

\[
f(x) = \begin{cases} 
3 & \text{if } x = a \\
2 & \text{if } x = b \\
1 & \text{if } x = c 
\end{cases}
\]

Then the function \(f\) is \((\gamma, \beta)\)-\(P_S\)-irresolute, but \(f\) is not \((\gamma, \beta)\)-\(P_S\)-continuous since \(\{2\}\) is a \(\beta\)-open set in \((Y, \sigma)\) containing \(f(b) = 2\), but there exist no \(\gamma\)-\(P_S\)-open set \(U\) in \((X, \tau)\) containing \(b\) such that \(f(U) \subseteq \{2\}\). By Remark 3.2, \(f\) is not \(\gamma\)-\(P_S\)-continuous.

**Example 3.6.** Consider the space \(X = \{a, b, c\}\) with the topologies \(\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}\) and \(\sigma = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}\). Define the operations \(\gamma\) and \(\beta\) on \(\tau\) and \(\sigma\) respectively as follows: For every \(A \in \tau\), \(\gamma(A) = A\) and for every \(B \in \sigma\)

\[
\beta(B) = \begin{cases} 
B & \text{if } c \in B \\
\text{Cl}(B) & \text{if } c \notin B 
\end{cases}
\]

Obviously, \(\tau_\gamma = \tau = \tau_{\gamma-P_S}(X)\), \(\sigma_\beta = \{\phi, X, \{c\}, \{a, b\}, \{b, c\}\}\) and \(\sigma_{\beta-P_S}(X) = \{\phi, X, \{c\}, \{a, b\}, \{a, c\}\}\).

Define a function \(f: (X, \tau) \to (X, \sigma)\) as follows:

\[
f(x) = \begin{cases} 
b & \text{if } x \in \{a, c\} \\
a & \text{if } x = b 
\end{cases}
\]

So, the function \(f\) is both \((\gamma, \beta)\)-\(P_S\)-continuous and \(\gamma\)-\(P_S\)-continuous, but \(f\) is not \((\gamma, \beta)\)-\(P_S\)-irresolute since \(\{a, c\}\) is a \(\beta\)-\(P_S\)-open set in \((X, \sigma)\) containing \(f(b) = a\), there exist no \(\gamma\)-\(P_S\)-open set \(U\) in \((X, \tau)\) containing \(c\) such that \(f(U) \subseteq \{a, c\}\).
4. Characterizations

We start with the most important characterizations of \((\gamma, \beta)\)-\(P_S\)-irresolute functions.

**Theorem 4.1.** For any function \(f: (X, \tau) \rightarrow (Y, \sigma)\). The following properties of \(f\) are equivalent:

1. \(f\) is \((\gamma, \beta)\)-\(P_S\)-irresolute.
2. The inverse image of every \(\beta\)-\(P_S\)-open set of \(Y\) is \(\gamma\)-\(P_S\)-open set in \(X\).
3. The inverse image of every \(\beta\)-\(P_S\)-closed set of \(Y\) is \(\gamma\)-\(P_S\)-closed set in \(X\).
4. \(f(\tau_\gamma P_S Cl(A)) \subseteq \sigma_\beta P_S Cl(f(A))\), for every subset \(A\) of \(X\).
5. \(\tau_\gamma P_S Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta P_S Cl(B))\), for every subset \(B\) of \(Y\).
6. \(f^{-1}(\sigma_\beta P_S Int(B)) \subseteq \tau_\gamma P_S Int(f^{-1}(B))\), for every subset \(B\) of \(Y\).
7. \(\sigma_\beta P_S Int(f(A)) \subseteq f(\tau_\gamma P_S Int(A))\), for every subset \(A\) of \(X\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(V\) be any \(\beta\)-\(P_S\)-open set in \(Y\). We have to show that \(f^{-1}(V)\) is \(\gamma\)-\(P_S\)-open set in \(X\). Let \(x \in f^{-1}(V)\). Then \(f(x) \in V\). By (1), there exists a \(\gamma\)-\(P_S\)-open set \(U\) of \(X\) containing \(x\) such that \(f(U) \subseteq V\). Which implies that \(x \in U \subseteq f^{-1}(V)\). Therefore, \(f^{-1}(V)\) is \(\gamma\)-\(P_S\)-open set in \(X\).

(2) \(\Rightarrow\) (3) Let \(F\) be any \(\beta\)-\(P_S\)-closed set of \(Y\). Then \(Y \setminus F\) is a \(\beta\)-\(P_S\)-open set of \(Y\). By (2), \(f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)\) is \(\gamma\)-\(P_S\)-open set in \(X\) and hence \(f^{-1}(F)\) is \(\gamma\)-\(P_S\)-closed set in \(X\).

(3) \(\Rightarrow\) (4) Let \(A\) be any subset of \(X\). Then \(f(A) \subseteq \sigma_\beta P_S Cl(f(A))\) and hence \(A \subseteq f^{-1}(\sigma_\beta P_S Cl(f(A)))\). Since \(\sigma_\beta P_S Cl(f(A))\) is \(\beta\)-\(P_S\)-closed set in \(Y\). Then by (3), we have \(f^{-1}(\sigma_\beta P_S Cl(f(A)))\) is \(\gamma\)-\(P_S\)-closed set in \(X\). Therefore, \(\tau_\gamma P_S Cl(A) \subseteq f^{-1}(\sigma_\beta P_S Cl(f(A)))\). Hence \(f(\tau_\gamma P_S Cl(A)) \subseteq \sigma_\beta P_S Cl(f(A))\).

(4) \(\Rightarrow\) (5) Let \(B\) be any subset of \(Y\). Then \(f^{-1}(B)\) is a subset of \(X\). By (4), we have \(f(\tau_\gamma P_S Cl(f^{-1}(B))) \subseteq \sigma_\beta P_S Cl(f(f^{-1}(B)))\) = \(\sigma_\beta P_S Cl(B)\). Hence \(\tau_\gamma P_S Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta P_S Cl(B))\).

(5) \(\Leftrightarrow\) (6) Let \(B\) be any subset of \(Y\). Then apply (5) to \(Y \setminus B\) we obtain \(\tau_\gamma P_S Cl(f^{-1}(Y \setminus B)) \subseteq f^{-1}(\sigma_\beta P_S Cl(Y \setminus B)) \Leftrightarrow \tau_\gamma P_S Cl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus \sigma_\beta P_S Int(B)) \Leftrightarrow X \setminus \tau_\gamma P_S Int(f^{-1}(B)) \subseteq X \setminus f^{-1}(\sigma_\beta P_S Int(B)) \Leftrightarrow f^{-1}(\sigma_\beta P_S Int(B)) \subseteq \tau_\gamma P_S Int(f^{-1}(B))\). Therefore, \(f^{-1}(\sigma_\beta P_S Int(B)) \subseteq \tau_\gamma P_S Int(f^{-1}(B))\).
(6) ⇒ (7) Let \( A \) be any subset of \( X \). Then \( f(A) \) is a subset of \( Y \). By (6), we have \( f^{-1}(\sigma_{\beta}P_S \text{Int}(f(A))) \subseteq \tau_\gamma P_S \text{Int}(f^{-1}(f(A))) = \tau_\gamma P_S \text{Int}(A) \). Therefore, \( \sigma_{\beta}P_S \text{Int}(f(A)) \subseteq f(\tau_\gamma P_S \text{Int}(A)). \)

(7) ⇒ (1) Let \( x \in X \) and let \( V \) be any \( \beta \)-\( P_S \)-open set of \( Y \) containing \( f(x) \). Then \( x \in f^{-1}(V) \) and \( f^{-1}(V) \) is a subset of \( X \). By (7), we have \( \sigma_{\beta}P_S \text{Int}(f(f^{-1}(V))) \subseteq f(\tau_\gamma P_S \text{Int}(f^{-1}(V))). \) Then \( \sigma_{\beta}P_S \text{Int}(V) \subseteq f(\tau_\gamma P_S \text{Int}(f^{-1}(V))) \). Since \( V \) is a \( \beta \)-\( P_S \)-open set. Then \( V \subseteq f(\tau_\gamma P_S \text{Int}(f^{-1}(V))) \) implies that \( f^{-1}(V) \subseteq \tau_\gamma P_S \text{Int}(f^{-1}(V)). \) Therefore, \( f^{-1}(V) \) is \( \gamma \)-\( P_S \)-open set in \( X \) which contains \( x \) and clearly \( f(f^{-1}(V)) \subseteq V \). Hence \( f \) is \( (\gamma, \beta) \)-\( P_S \)-irresolute function.

**Theorem 4.2.** Let \( f: (X, \tau) \to (Y, \sigma) \) be any function. Then the following statements are equivalent:

1. \( f \) is \((\gamma, \beta) \)-\( P_S \)-continuous.
2. \( f^{-1}(V) \) is \( \gamma \)-\( P_S \)-open set in \( X \), for every \( \beta \)-open set \( V \) in \( Y \).
3. \( f^{-1}(F) \) is \( \gamma \)-\( P_S \)-closed set in \( X \), for every \( \beta \)-closed set \( F \) in \( Y \).
4. \( f(\tau_\gamma P_S \text{Cl}(A)) \subseteq \sigma_{\beta} \text{Cl}(f(A)) \), for every subset \( A \) of \( X \).
5. \( \sigma_{\beta} \text{Int}(f(A)) \subseteq f(\tau_\gamma P_S \text{Int}(A)) \), for every subset \( A \) of \( X \).
6. \( f^{-1}(\sigma_{\beta} \text{Int}(B)) \subseteq \tau_\gamma P_S \text{Int}(f^{-1}(B)) \), for every subset \( B \) of \( Y \).
7. \( \tau_\gamma P_S \text{Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_{\beta} \text{Cl}(B)) \), for every subset \( B \) of \( Y \).

**Proof.** Similar to Theorem 4.1 and hence it is ommited.

**Theorem 4.3.** The following properties are equivalent for any function \( f: (X, \tau) \to (Y, \sigma) \):

1. \( f \) is \((\gamma, \beta) \)-\( P_S \)-irresolute.
2. For every \( x \in X \) and for every \( \beta \)-\( P_S \)-neighbourhood \( N \) of \( Y \) such that \( f(x) \in N \), there exists a \( \gamma \)-\( P_S \)-neighbourhood \( M \) of \( X \) such that \( x \in M \) and \( f(M) \subseteq N \).
3. The inverse image of every \( \beta \)-\( P_S \)-neighbourhood of \( f(x) \) is \( \gamma \)-\( P_S \)-neighbourhood of \( x \in X \).

**Proof.** It is clear and hence it is ommited.
Lemma 4.4. Let \( f : (X, \tau) \to (Y, \sigma) \) be a \( \gamma \)-continuous and \( \beta \)-open function, then the following statements are true:

1. If \( V \) is \( \beta \)-preopen set of \( Y \), then \( f^{-1}(V) \) is \( \gamma \)-preopen set in \( X \).
2. If \( F \) is \( \beta \)-semiclosed set of \( Y \), then \( f^{-1}(F) \) is \( \gamma \)-semiclosed set in \( X \).

**Proof.** Let \( V \) be a \( \beta \)-\( P_S \)-open set of \( Y \), then \( V \) is a \( \beta \)-preopen set of \( Y \) and \( V = \bigcup_{i \in I} F_i \) where \( F_i \) is \( \beta \)-semiclosed set in \( Y \) for each \( i \). Then \( f^{-1}(V) = f^{-1}(\bigcup_{i \in I} F_i) = \bigcup_{i \in I} f^{-1}(F_i) \) where \( F_i \) is \( \beta \)-semiclosed set in \( Y \) for each \( i \). Since \( f \) is a \( \gamma \)-continuous and \( \beta \)-open function. Then by Lemma 4.4 (1), \( f^{-1}(V) \) is \( \gamma \)-preopen set of \( X \) and by Lemma 4.4 (2), \( f^{-1}(F_i) \) is \( \gamma \)-semiclosed set of \( X \) for each \( i \). Hence by Lemma 2.3, \( f^{-1}(V) \) is \( \gamma \)-\( P_S \)-open set in \( X \). \( \Box \)

Corollary 4.6. If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \gamma \)-continuous and \( \beta \)-open function and \( F \) is \( \beta \)-\( P_S \)-closed set of \( Y \), then \( f^{-1}(F) \) is \( \gamma \)-\( P_S \)-closed set in \( X \).

Lemma 4.7. If a function \( f : (X, \tau) \to (Y, \sigma) \) is both \( \gamma \)-continuous and \( \beta \)-open, then \( f \) is \( (\gamma, \beta) \)-\( P_S \)-irresolute.

**Proof.** The proof follows directly from Lemma 4.5 and Theorem 4.1. \( \Box \)

Some other characterizations of \((\gamma, \beta)\)-\( P_S \)-irresolute functions are mentioned in the following.

Theorem 4.8. Let \( f : (X, \tau) \to (Y, \sigma) \) be any function. Then the following properties are equivalent:

1. \( f \) is \( (\gamma, \beta) \)-\( P_S \)-irresolute.
2. \( \tau_{\gamma-P_S Bd}(f^{-1}(B)) \subseteq f^{-1}(\sigma_{\beta-P_S Bd}(B)) \), for each subset \( B \) of \( Y \).
3. \( f(\tau_{\gamma-P_S Bd}(A)) \subseteq \sigma_{\beta-P_S Bd}(f(A)) \), for each subset \( A \) of \( X \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( f \) be a \((\gamma, \beta)\)-\( P_S \)-irresolute function and \( B \) be any subset of \((Y, \sigma)\). Then by Theorem 4.1 (2) and (5), we have \( \tau_{\gamma-P_S Bd}(f^{-1}(B)) = \tau_{\gamma-P_S Cl}(f^{-1}(B)) \setminus \tau_{\gamma-P_S Int}(f^{-1}(B)) \subseteq f^{-1}(\sigma_{\beta-P_S Cl}(B)) \setminus f^{-1}(\sigma_{\beta-P_S Int}(B)) \subseteq f^{-1}(\sigma_{\beta-P_S Cl}(B)) \setminus f^{-1}(\sigma_{\beta-P_S Int}(B)) = f^{-1}(\sigma_{\beta-P_S Cl}(B)) \setminus f^{-1}(\sigma_{\beta-P_S Int}(B)) = f^{-1}(\sigma_{\beta-P_S Bd}(B)) \). Therefore, \( \tau_{\gamma-P_S Bd}(f^{-1}(B)) \subseteq f^{-1}(\sigma_{\beta-P_S Bd}(B)) \).

(2) \( \Rightarrow \) (3). Let \( A \) be any subset of \( X \). Then \( f(A) \) is a subset of \( Y \). Then by (2), we have \( \tau_{\gamma-P_S Bd}(f^{-1}(f(A))) \subseteq f^{-1}(\sigma_{\beta-P_S Bd}(f(A))) \) implies
that $\tau_\gamma P_S Bd(A) \subseteq f^{-1}(\sigma_\beta P_S Bd(f(A)))$ and hence $f(\tau_\gamma P_S Bd(A)) \subseteq \sigma_\beta P_S Bd(f(A))$. This completes the proof.

(3) $\Rightarrow$ (1). Let $E$ be any $\beta$-$P_S$-closed set in $Y$. Then $f^{-1}(E)$ is a subset of $X$. So by using part (3), we have $f(\tau_\gamma P_S Bd(f^{-1}(E))) \subseteq \sigma_\beta P_S Bd(f^{-1}(E))) = \sigma_\beta P_S Bd(E)$ implies that $f(\tau_\gamma P_S Bd(f^{-1}(E))) \subseteq \sigma_\beta P_S Bd(E) \subseteq \sigma_\beta P_S Cl(E) = E$ and hence $f(\tau_\gamma P_S Bd(f^{-1}(E))) \subseteq E$. This implies that $\tau_\gamma P_S Bd(f^{-1}(E)) \subseteq f^{-1}(E)$. Thus, by Lemma 2.6 (5), $f^{-1}(E)$ is $\gamma$-$P_S$-closed set in $X$. Consequently by Theorem 4.1, $f$ is $($\gamma, $\beta)$-$P_S$-irresolute function.

**Theorem 4.9.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be any function. Then the following properties are equivalent:

1. $f$ is $($\gamma, $\beta)$-$P_S$-continuous.
2. For each subset $A$ in $X$, $f(\tau_\gamma P_S Bd(A)) \subseteq \sigma_\beta Bd(f(A))$.
3. For each subset $B$ in $Y$, $\tau_\gamma P_S Bd(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta Bd(B))$.

**Proof.** The proof is similar to Theorem 4.8, and hence it is omitted.

**Theorem 4.10.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be any function. Then the following properties are equivalent:

1. $f$ is $($\gamma, $\beta)$-$P_S$-irresolute.
2. $f(\tau_\gamma P_S D(A)) \subseteq \sigma_\beta P_S Cl(f(A))$, for each subset $A$ of $X$.
3. $\tau_\gamma P_S D(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta P_S Cl(B))$, for each subset $B$ of $Y$.

**Proof.** (1) $\Rightarrow$ (2). Let $f$ be a $($\gamma, $\beta)$-$P_S$-continuous function and $A$ be any subset of $X$. Then by Theorem 4.1 (4), we have $f(\tau_\gamma P_S Cl(A)) \subseteq \sigma_\beta P_S Cl(f(A))$. Then by Lemma 2.6 (6), we obtain $f(\tau_\gamma P_S D(A)) \subseteq f(\tau_\gamma P_S Cl(A))$ which implies that $f(\tau_\gamma P_S D(A)) \subseteq \sigma_\beta P_S Cl(f(A))$.

(2) $\Rightarrow$ (3). Let $B$ be any subset of $Y$. Then $f^{-1}(B)$ is a subset of $X$. Then by hypothesis, we get $f(\tau_\gamma P_S D(f^{-1}(B))) \subseteq \sigma_\beta P_S Cl(f(f^{-1}(B))) = \sigma_\beta P_S Cl(B)$ and hence $\tau_\gamma P_S D(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta P_S Cl(B))$. This completes the proof.

(3) $\Rightarrow$ (1). Let $F$ be any $\beta$-$P_S$-closed set in $Y$. Then by (3), we have $\tau_\gamma P_S D(f^{-1}(F)) \subseteq f^{-1}(\sigma_\beta P_S Cl(F)) = f^{-1}(F)$ and hence $\tau_\gamma P_S D(f^{-1}(F)) \subseteq f^{-1}(F)$. So by Lemma 2.6 (7), we get $f^{-1}(F)$ is $\gamma$-$P_S$-closed set in $X$. Therefore, by Theorem 4.1, $f$ is $($\gamma, $\beta)$-$P_S$-irresolute function.

**Theorem 4.11.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be any function. Then the following properties are equivalent:
1. $f$ is $(\gamma, \beta)$-$P_S$-continuous.

2. $f(\tau_\gamma P_SD(A)) \subseteq \sigma_\beta Cl(f(A))$, for each subset $A$ of $X$.

3. $\tau_\gamma P_SD(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta Cl(B))$, for each subset $B$ of $Y$.

Proof. The proof is similar to Theorem 4.10, and hence it is omitted. \hfill $\square$

**Proposition 4.12.** If a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is $(\gamma, \beta)$-$P_S$-irresolute, then for each $x \in X$ and each $\beta$-$P_S$-open set $V$ of $Y$ containing $f(x)$, there exists a $\gamma$-semiclosed set $F$ in $X$ such that $x \in F$ and $f(F) \subseteq V$. Furthermore, if $f$ is $(\gamma, \beta)$-precontinuous, then the converse also holds.

Proof. Suppose $f$ be a $(\gamma, \beta)$-$P_S$-irresolute function and let $V$ be any $\beta$-$P_S$-open set of $Y$ such that $f(x) \in V$, for each $x \in X$. Then there exists a $\gamma$-$P_S$-open set $U$ of $X$ such that $x \in U$ and $f(U) \subseteq V$. Since $U$ is $\gamma$-$P_S$-open set. Then for each $x \in U$, there exists a $\gamma$-semiclosed set $F$ of $X$ such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$.

Now suppose that $f$ is $(\gamma, \beta)$-precontinuous function. Let $V$ be any $\beta$-$P_S$-open set of $Y$. We have to show that $f^{-1}(V)$ is $\gamma$-$P_S$-open set in $X$. Since every $\beta$-$P_S$-open set is $\beta$-preopen, then $f^{-1}(V)$ is $\gamma$-preopen set in $X$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, there exists a $\gamma$-semiclosed set $F$ of $X$ containing $x$ such that $f(F) \subseteq V$. Which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, by Definition 2.2, $f^{-1}(V)$ is $\gamma$-$P_S$-open set in $X$. Hence by Theorem 4.1, $f$ is $(\gamma, \beta)$-$P_S$-irresolute. This completes the proof. \hfill $\square$

5. Properties

**Theorem 5.1.** Let $(X, \tau)$ be $\gamma$-semi$T_1$ space and $(Y, \sigma)$ be $\beta$-semi$T_1$ space. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is $(\gamma, \beta)$-$P_S$-irresolute if and only if $f$ is $(\gamma, \beta)$-precontinuous.

Proof. This is an immediate consequence of Theorem 2.9 (2). \hfill $\square$

**Theorem 5.2.** Let $(Y, \sigma)$ be $\beta$-locally indiscrete space. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is $(\gamma, \beta)$-$P_S$-irresolute if and only if $f$ is $(\gamma, \beta)$-$P_S$-continuous.

Proof. Follows directly from Theorem 2.9 (1). \hfill $\square$

**Theorem 5.3.** Let $(Y, \sigma)$ be $\beta$-regular space. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\gamma$-$P_S$-continuous if and only if $f$ is $(\gamma, \beta)$-$P_S$-continuous.
Proof. Follows directly from Theorem 2.9 (5).

**Theorem 5.4.** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function. If \( (Y, \sigma) \) is \( \beta \)-hyperconnected space, then \( f \) is \( (\gamma, \beta)P_S \)-irresolute.

Proof. This is an immediate consequence of Theorem 2.9 (3).

**Proposition 5.5.** If a function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( (\gamma, \beta)P_S \)-irresolute (resp. \( (\gamma, \beta)P_S \)-continuous), then the following properties are true:

1. for each \( x \in X \) and each \( \beta \)-P\_S-open (resp. \( \beta \)-open) set \( V \) of \( Y \) such that \( f(x) \in V \), there exists a \( \gamma \)-preopen set \( U \) of \( X \) such that \( x \in U \) and \( f(U) \subseteq V \).

2. for each \( x \in X \) and each \( \beta \)-regular-open set \( V \) of \( Y \) containing \( f(x) \), there exists a \( \gamma \)-P\_S-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \).

Proof. 1) Since every \( \gamma \)-P\_S-open set of \( X \) is \( \gamma \)-preopen, then by using this in Definition 3.1 we get the proof.

2) Since every \( \beta \)-regular-open set of \( Y \) is both \( \beta \)-open and \( \beta \)-P\_S-open, then the proof follows directly from Definition 3.1.

**Proposition 5.6.** A function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( (\gamma, \beta)P_S \)-irresolute (resp. \( (\gamma, \beta)P_S \)-continuous), if the following properties are true:

1. for each \( x \in X \) and each \( \beta \)-preopen set \( V \) of \( Y \) such that \( f(x) \in V \), there exists a \( \gamma \)-P\_S-open set \( U \) of \( X \) such that \( x \in U \) and \( f(U) \subseteq V \).

2. for each \( x \in X \) and each \( \beta \)-P\_S-open (resp. \( \beta \)-open) set \( V \) of \( Y \) containing \( f(x) \), there exists a \( \gamma \)-regular-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \).

Proof. 1) The proof is clear since every \( \beta \)-P\_S-open (resp. \( \beta \)-open) set of \( Y \) is \( \beta \)-preopen.

2) Obvious since every \( \gamma \)-regular-open set of \( X \) is \( \gamma \)-P\_S-open and hence it is omitted.

**Theorem 5.7.** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be any function, then: \( X\setminus\tau_\gamma-P_SC(f) = \bigcup\{\tau_\gamma-P_SBd(f^{-1}(V)) : V \text{ is a } \beta-P_S\text{-open in } (Y, \sigma) \text{ such that } f(x) \in V \text{ for each } x \in X\} \), where \( \tau_\gamma-P_SC(f) \) denotes the set of points at which \( f \) is \( (\gamma, \beta)P_S \)-irresolute function.
Proof. Let \( x \in \tau_{\gamma} P_{S} C(f) \). Then there exists \( \beta-P_{S} \)-open set \( V \) in \( (Y, \sigma) \) containing \( f(x) \) such that \( f(U) \not\subseteq V \) for every \( \gamma-P_{S} \)-open set \( U \) of \( (X, \tau) \) containing \( x \). Hence \( U \cap X \setminus f^{-1}(V) \neq \emptyset \) for every \( \gamma-P_{S} \)-open set \( U \) of \( (X, \tau) \) containing \( x \). Therefore, \( x \in \tau_{\gamma} P_{S} C(X \setminus f^{-1}(V)) \). Then \( x \in f^{-1}(V) \cap \tau_{\gamma} P_{S} C(X \setminus f^{-1}(V)) \subseteq \tau_{\gamma} P_{S} C(f^{-1}(V)) \cap \tau_{\gamma} P_{S} C(X \setminus f^{-1}(V)) = \tau_{\gamma} P_{S} \text{Bd}(f^{-1}(V)). \) Then \( X \setminus \tau_{\gamma} P_{S} C(f) \subseteq \cup \{ \tau_{\gamma} P_{S} \text{Bd}(f^{-1}(V)) : V \text{ is } \beta-P_{S} \)-open in \( (Y, \sigma) \) such that \( f(x) \in V \) for each \( x \in X \}. \)

Conversely, let \( x \notin X \setminus \tau_{\gamma} P_{S} C(f) \). Then for each \( \beta-P_{S} \)-open set \( V \) in \( (Y, \sigma) \) containing \( f(x), f^{-1}(V) \) is \( \gamma-P_{S} \)-open set of \( (X, \tau) \) containing \( x \). So \( x \in \tau_{\gamma} P_{S} \text{Int}(f^{-1}(V)) \) and hence \( x \notin \tau_{\gamma} P_{S} \text{Bd}(f^{-1}(V)) \) for every \( \beta-P_{S} \)-open set \( V \) in \( (Y, \sigma) \) containing \( f(x) \). Therefore, \( X \setminus \tau_{\gamma} P_{S} C(f) \supseteq \cup \{ \tau_{\gamma} P_{S} \text{Bd}(f^{-1}(V)) : V \text{ is } \beta-P_{S} \)-open in \( (Y, \sigma) \) such that \( f(x) \in V \) for each \( x \in X \}. \)

The proof of the following theorem is similar to Theorem 5.7 and is thus omitted.

**Theorem 5.8.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be any function, then: \( X \setminus \tau_{\gamma} P_{S} C(f) = \cup \{ \tau_{\gamma} P_{S} \text{Bd}(f^{-1}(V)) : V \text{ is } \beta \)-open in \( (Y, \sigma) \) such that \( f(x) \in V \) for each \( x \in X \}, \) where \( \tau_{\gamma} P_{S} C(f) \) denotes the set of points at which \( f \) is \( (\gamma, \beta) \)-\( P_{S} \)-continuous function.

Now, we will define more types of \( \gamma-P_{S} \)-functions by using \( \gamma-P_{S} \)-open set which are defined as follows.

**Definition 5.9.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called \( (\gamma, \beta) \)-\( P_{S} \)-open (resp. \( (\gamma, \beta P_{S}) \)-open) if for every \( \gamma-P_{S} \)-open (resp. \( \gamma \)-open) set \( V \) of \( X \), \( f(V) \) is \( \beta-P_{S} \)-open set in \( Y \).

**Definition 5.10.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called \( (\gamma, \beta) \)-\( P_{S} \)-closed (resp. \( (\gamma, \beta P_{S}) \)-open) if for every \( \gamma-P_{S} \)-closed (\( \gamma \)-closed) set \( F \) of \( X \), \( f(F) \) is \( \beta-P_{S} \)-closed set in \( Y \).

**Theorem 5.11.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( (\gamma, \beta) \)-\( P_{S} \)-open if and only if for every \( x \in X \) and for every \( \gamma-P_{S} \)-neighbourhood \( N \) of \( x \), there exists a \( \beta-P_{S} \)-neighbourhood \( M \) of \( Y \) such that \( f(x) \in M \) and \( M \subseteq f(N) \).

Proof. Obvious.

**Theorem 5.12.** The following statements are equivalent for a function \( f : (X, \tau) \rightarrow (Y, \sigma) \):

1. \( f \) is \( (\gamma, \beta) \)-\( P_{S} \)-open.
2. \( f(\tau_{\gamma} P_{S} \text{Int}(A)) \subseteq \sigma_{P_{S}} \text{Int}(f(A)) \), for every \( A \subseteq X \).


3. \( \tau_\gamma P_S \text{Int}(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta P_S \text{Int}(B)) \), for every \( B \subseteq Y \).

**Proof.** The proof is similar to Theorem 4.1.

**Theorem 5.13.** The following properties of \( f \) are equivalent for a function \( f: (X, \tau) \rightarrow (Y, \sigma) \):

1. \( f \) is \((\gamma, \beta)P_S\)-closed.

2. \( f^{-1}(\sigma_\beta P_S \text{Cl}(B)) \subseteq \tau_\gamma P_S \text{Cl}(f^{-1}(B)) \), for every \( B \subseteq Y \).

3. \( \sigma_\beta P_S \text{Cl}(f(A)) \subseteq f(\tau_\gamma P_S \text{Cl}(A)) \), for every \( A \subseteq X \).

4. \( \sigma_\beta P_S D(f(A)) \subseteq f(\tau_\gamma P_S \text{Cl}(A)) \), for every \( A \subseteq X \).

**Proof.** The proof is similar to Theorem 4.1.

**Definition 5.14.** Let \( id: \tau \rightarrow P(X) \) be the identity operation. If \( f \) is \((id, \beta)P_S\)-closed, then for every \( \gamma P_S\)-closed set \( F \) of \( X \), \( f(F) \) is \( \beta P_S\)-closed set in \( Y \).

**Theorem 5.15.** If a function \( f \) is bijective and \( f^{-1}: (Y, \sigma) \rightarrow (X, \tau) \) is \((id, \beta)P_S\)-irresolute, then \( f \) is \((id, \beta)P_S\)-closed.

**Proof.** Follows from Definition 5.14 and Definition 3.1.

**Theorem 5.16.** Suppose that a function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is both \((\gamma, \beta)P_S\)-irresolute and \((\gamma, \beta)P_S\)-closed, then:

1. For every \( \gamma P_S\)-g-closed set \( A \) of \( (X, \tau) \), the image \( f(A) \) is \( \beta P_S\)-g-closed in \( (Y, \sigma) \).

2. For every \( \beta P_S\)-g-closed set \( B \) of \( (Y, \sigma) \) the inverse set \( f^{-1}(B) \) is \( \gamma P_S\)-g-closed in \( (X, \tau) \).

**Proof.** (1) Let \( G \) be any \( \beta P_S\)-open set in \( (Y, \sigma) \) such that \( f(A) \subseteq G \). Since \( f \) is \((\gamma, \beta)P_S\)-irresolute function, then by Theorem 4.1 (2), \( f^{-1}(G) \) is \( \gamma P_S\)-open set in \( (X, \tau) \). Since \( A \) is \( \gamma P_S\)-g-closed and \( A \subseteq f^{-1}(G) \), we have \( \tau_\gamma P_S \text{Cl}(A) \subseteq f^{-1}(G) \), and hence \( f(\tau_\gamma P_S \text{Cl}(A)) \subseteq G \). Since \( \tau_\gamma P_S \text{Cl}(A) \) is \( \gamma P_S\)-closed set and \( f \) is \((\gamma, \beta)P_S\)-closed, then \( f(\tau_\gamma P_S \text{Cl}(A)) \) is \( \beta P_S\)-closed set in \( Y \). Therefore, \( \sigma_\beta P_S \text{Cl}(f(A)) \subseteq \sigma_\beta P_S \text{Cl}(f(\tau_\gamma P_S \text{Cl}(A))) = f(\tau_\gamma P_S \text{Cl}(A)) \subseteq G \). This implies that \( f(A) \) is \( \beta P_S\)-g-closed in \( (Y, \sigma) \).

(2) Let \( H \) be any \( \gamma P_S\)-open set of \( (X, \tau) \) such that \( f^{-1}(B) \subseteq H \). Let \( C = \tau_\gamma P_S \text{Cl}(f^{-1}(B)) \cap (X \setminus H) \), then \( C \) is \( \gamma P_S\)-closed set in \( (X, \tau) \). Since \( f \) is
Therefore, \( f \) is \( \beta\gamma \)-\( P_S \)-closed function. Then \( f(C) \) is \( \beta\gamma \)-\( P_S \)-closed in \((Y, \sigma)\). Since \( f \) is \((\gamma, \beta)\)-\( P_S \)-irresolute function, then by using Theorem 4.1 (4), we have \( f(C) = f(\tau\gamma\beta P_S Cl(f^{-1}(B))) \cap f(X \setminus H) \subseteq \sigma\gamma\beta P_S Cl(B) \cap f(X \setminus H) \subseteq \sigma\gamma\beta P_S Cl(B) \cap (Y \setminus B) \). This implies that \( f(C) = \phi \), and hence \( C = \phi \). So \( \tau\gamma\beta P_S Cl(f^{-1}(B)) \subseteq H \). Therefore, \( f^{-1}(B) \) is \( \gamma\beta P_S -g\)-closed in \((X, \tau)\).

**Theorem 5.17.** Let \( f : (X, \tau) \to (Y, \sigma) \) be an injective, \((\gamma, \beta)\)-\( P_S \)-irresolute and \((\gamma, \beta)\)-\( P_S \)-closed function. If \((Y, \sigma)\) is \( \beta\gamma\beta P_S \)-\( T_{\frac{1}{2}} \), then \((X, \tau)\) is \( \gamma\gamma\beta P_S \)-\( T_{\frac{1}{2}} \).

**Proof.** Let \( G \) be any \( \gamma\gamma\beta P_S -g\)-closed set of \((X, \tau)\). Since \( f \) is \((\gamma, \beta)\)-\( P_S \)-irresolute and \((\gamma, \beta)\)-\( P_S \)-closed function. Then by Theorem 5.16 (1), \( f(G) \) is \( \beta\gamma\gamma P_S \)-\( g\)-closed in \((Y, \sigma)\). Since \((Y, \sigma)\) is \( \beta\gamma\gamma P_S \)-\( T_{\frac{1}{2}} \), then \( f(G) \) is \( \beta\gamma\gamma P_S \)-closed in \( Y \). Again, since \( f \) is \((\gamma, \beta)\)-\( P_S \)-irresolute, then by Theorem 4.1, \( f^{-1}(f(G)) \) is \( \gamma\gamma\beta P_S \)-closed in \( X \). Hence \( G \) is \( \gamma\gamma\beta P_S \)-closed in \( X \) since \( f \) is injective. Therefore, a space \((X, \tau)\) is \( \gamma\gamma P_S \)-\( T_{\frac{1}{2}} \).

**Theorem 5.18.** Let a function \( f : (X, \tau) \to (Y, \sigma) \) be a surjective, \((\gamma, \beta)\)-\( P_S \)-irresolute and \((\gamma, \beta)\)-\( P_S \)-closed function. If \((X, \tau)\) is \( \gamma\gamma P_S \)-\( T_{\frac{1}{2}} \), then \((Y, \sigma)\) is \( \beta\gamma P_S \)-\( T_{\frac{1}{2}} \).

**Proof.** Let \( H \) be a \( \beta P_S \)-\( g\)-closed set of \((Y, \sigma)\). Since a function \( f \) is \((\gamma, \beta)\)-\( P_S \)-irresolute and \((\gamma, \beta)\)-\( P_S \)-closed function. Then by Theorem 5.16 (2), \( f^{-1}(H) \) is \( \gamma\gamma P_S \)-\( g\)-closed in \((X, \tau)\). Since \((X, \tau)\) is \( \gamma\gamma P_S \)-\( T_{\frac{1}{2}} \), then we have, \( f^{-1}(H) \) is \( \gamma\gamma P_S \)-closed set in \( X \). Again, since \( f \) is \((\gamma, \beta)\)-\( P_S \)-closed function, then \( f(f^{-1}(H)) \) is \( \beta\gamma P_S \)-closed in \( Y \). Therefore, \( H \) is \( \beta\gamma P_S \)-closed in \( Y \) since \( f \) is surjective. Hence \((Y, \sigma)\) is \( \beta\gamma P_S \)-\( T_{\frac{1}{2}} \) space.

**Remark 5.19.** Every \( \beta P_S \)-open (resp., \( \beta P_S \)-closed) function is \((\gamma, \beta P_S)\)-open (resp., \((\gamma, \beta P_S)\)-closed), but the converse is not true as it is shown in the following example.

**Example 5.20.** Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\phi, \{c\}, \{b, c\}, X\} \) and \( Y = \{1, 2, 3\} \) with the topology \( \sigma = \{\phi, Y, \{2\}, \{1, 3\}\} \). Define operations \( \beta \) on \( \sigma \) by \( \beta(B) = B \) for all \( B \in \sigma \) and \( \gamma \) on \( \tau \) as follows: For every \( A \in \tau \)

\[
\gamma(A) = \begin{cases} 
A & \text{if } A = \{c\} \\
X & \text{otherwise}
\end{cases}
\]

Then \( \tau_\gamma = \{\phi, X, \{c\}\} \) and hence \( \tau_\gamma P_S O(X) = \{\phi, X\} \).

Let \( f : (X, \tau) \to (Y, \sigma) \) be a function defined as follows:

\[
f(x) = \begin{cases} 
3 & \text{if } x = a \\
1 & \text{if } x = b \\
2 & \text{if } x = c
\end{cases}
\]
So $f$ is $(\gamma, \beta P_S)$-open (resp., $(\gamma, \beta P_S)$-closed) function, but $f$ is not $\beta P_S$-open (resp., $\beta P_S$-closed) since $\{c\} \in \tau$, but $f(\{b, c\}) = \{1, 2\}$ is not $\beta P_S$-open set in $(Y, \sigma)$. Again since $\{a\}$ is closed set in $(X, \tau)$, but $f(\{a\}) = \{3\}$ is not $\beta P_S$-closed set in $(Y, \sigma)$.

**Theorem 5.21.** Let $(X, \tau)$ be $\gamma$-semi$T_1$ space and $(Y, \sigma)$ be $\beta$-semi$T_1$ space. A function $f: (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta)-P_S$-closed if and only if $f$ is $(\gamma, \beta)$-preclosed.

**Proof.** This is an immediate consequence of Theorem 2.9 (2).

**Theorem 5.22.** Let $(X, \tau)$ be $\gamma$-locally indiscrete space and $f: (X, \tau) \to (Y, \sigma)$ be a function, then the following properties of $f$ are equivalent:

1. $(\gamma, \beta)-P_S$-open.
2. $(\gamma, \beta)-P_S$-closed.
3. $(\gamma, \beta P_S)$-closed.
4. $(\gamma, \beta P_S)$-open.

**Proof.** Follows directly from Theorem 2.9 (1).

**Theorem 5.23.** Let $(X, \tau)$ be $\gamma$-regular space. A function $f: (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta P_S)$-open (resp., $(\gamma, \beta P_S)$-closed) if and only if $f$ is $\beta P_S$-open (resp., $\beta P_S$-closed).

**Proof.** This is an immediate consequence of Theorem 2.9 (5).

**Theorem 5.24.** Let $f: (X, \tau) \to (Y, \sigma)$ be a surjective function. If $(X, \tau)$ is $\gamma$-hyperconnected space, then $f$ is $(\gamma, \beta)-P_S$-open.

**Proof.** This is an immediate consequence of Theorem 2.9 (3).

**Theorem 5.25.** A function $f: (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta)-P_S$-closed if and only if for each subset $S$ of $Y$ and each $\gamma P_S$-open set $O$ in $X$ containing $f^{-1}(S)$, there exists a $\beta P_S$-open set $R$ in $Y$ containing $S$ such that $f^{-1}(R) \subseteq O$.

**Proof.** Suppose that $f$ is $(\gamma, \beta)-P_S$-closed function and let $O$ be a $\gamma P_S$-open set in $X$ containing $f^{-1}(S)$, where $S$ is any subset in $Y$. Then $f(X \setminus O)$ is $\beta P_S$-open set in $Y$. If we put $R = Y \setminus f(X \setminus O)$. Then $R$ is $\beta P_S$-closed set in $Y$ such that $S \subseteq R$ and $f^{-1}(R) \subseteq O$. 
Conversely, let $F$ be a $\gamma$-$P_S$-closed set in $X$. Let $S = Y \setminus f(F) \subseteq Y$. Then $f^{-1}(S) \subseteq X \setminus F$ and $X \setminus F$ is $\gamma$-$P_S$-open set in $X$. By hypothesis, there exists a $\beta$-$P_S$-open set $R$ in $Y$ such that $S = Y \setminus f(F) \subseteq R$ and $f^{-1}(R) \subseteq X \setminus F$. For $f^{-1}(R) \subseteq X \setminus F$ implies $R \subseteq f(X \setminus F) \subseteq Y \setminus f(F)$. Hence $R = Y \setminus f(F)$. Since $R$ is $\beta$-$P_S$-open set in $Y$. Then $f(F)$ is $\beta$-$P_S$-closed set in $Y$. Therefore, $f$ is $(\gamma, \beta)$-$P_S$-closed function.

**Theorem 5.26.** A function $f : (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta P_S)$-closed if and only if for each subset $S$ of $Y$ and each $\gamma$-open set $O$ in $X$ containing $f^{-1}(S)$, there exists a $\beta$-$P_S$-open set $R$ in $Y$ containing $S$ such that $f^{-1}(R) \subseteq O$.

**Proof.** The proof is similar to Theorem 5.25.

**Definition 5.27.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $(\gamma, \beta)$-$P_S$-homeomorphism, if $f$ is bijective, $(\gamma, \beta)$-$P_S$-irresolute and $f^{-1}$ is $(\gamma, \beta)$-$P_S$-irresolute.

**Theorem 5.28.** For a bijective function $f : (X, \tau) \to (Y, \sigma)$. The following properties of $f$ are equivalent:

1. $f^{-1}$ is $(\gamma, \beta)$-$P_S$-irresolute.
2. $f$ is $(\gamma, \beta)$-$P_S$-open.
3. $f$ is $(\gamma, \beta)$-$P_S$-closed.

**Proof.** Obvious.

**Theorem 5.29.** The following conditions of $f$ are equivalent for a bijective function $f : (X, \tau) \to (Y, \sigma)$:

1. $f$ is $(\gamma, \beta)$-$P_S$-homeomorphism.
2. $f$ is $(\gamma, \beta)$-$P_S$-irresolute and $(\gamma, \beta)$-$P_S$-open.
3. $f$ is $(\gamma, \beta)$-$P_S$-irresolute and $(\gamma, \beta)$-$P_S$-closed.
4. $f(\tau_\gamma P_S Cl(A)) = \sigma_\beta P_S Cl(f(A))$ for each subset $A$ of $X$.

**Proof.** Straightforward.

**Proposition 5.30.** Assume that a function $f : (X, \tau) \to (Y, \sigma)$ is a $(\gamma, \beta)$-$P_S$-homeomorphism. If $(X, \tau)$ is $\gamma$-$P_S$-$T_{\frac{1}{2}}$, then $(Y, \sigma)$ is $\beta$-$P_S$-$T_{\frac{1}{2}}$. 
Proof. Let \( \{y\} \) be any singleton set of \((Y, \sigma)\). Then there exists an element \( x \) of \( X \) such that \( y = f(x) \). So by hypothesis and Theorem 2.9 (4), we have \( \{x\} \) is \( \gamma - P_S \)-closed or \( \gamma - P_S \)-open set in \( X \). By using Theorem 4.1, \( \{y\} \) is \( \beta - P_S \)-closed or \( \beta - P_S \)-open set. Then by Theorem 2.9 (4), \((Y, \sigma)\) is \( \beta - P_S - T_{1/2} \) space. \( \square \)

In the end of this section, we shall obtain some conditions under which the composition of two functions is \((\gamma, \beta) - P_S\)- irresolute and \((\gamma, \beta) - P_S\)-continuous.

**Theorem 5.31.** Let \( \alpha \) be an operation on the topological space \((Z, \rho)\). Let \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \rho) \) be functions. Then the composition function \( g \circ f: (X, \tau) \to (Z, \rho) \) is \((\gamma, \alpha) - P_S\)-continuous if \( f \) and \( g \) satisfy one of the following conditions:

1. \( f \) is \((\gamma, \beta) - P_S\)-irresolute and \( g \) is \((\beta, \alpha) - P_S\)-continuous.

2. \( f \) is \((\gamma, \beta) - P_S\)-irresolute and \( g \) is \( \beta - P_S\)-continuous.

**Proof.**

(1) Let \( V \) be an \( \alpha \)-open subset of \((Z, \rho)\). Since \( g \) is \((\beta, \alpha) - P_S\)- continuous, then \( g^{-1}(V) \) is \( \beta - P_S\)-open in \((Y, \sigma)\). Since \( f \) is \((\gamma, \beta) - P_S\)- irresolute, then by Theorem 4.1, \((g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is \( \gamma - P_S\)-open in \( X \). Therefore, \( g \circ f \) is \((\gamma, \alpha) - P_S\)-continuous.

(2) The proof follows directly from the part (1) since every \( \beta - P_S\)-continuous is \((\beta, \alpha) - P_S\)-continuous. \( \square \)

**Theorem 5.32.** Let \( \alpha \) be an operation on the topological space \((Z, \rho)\). If the functions \( f: (X, \tau) \to (Y, \sigma) \) is \((\gamma, \beta) - P_S\)- irresolute and \( g: (Y, \sigma) \to (Z, \rho) \) is \((\beta, \alpha) - P_S\)- irresolute. Then \( g \circ f: (X, \tau) \to (Z, \rho) \) is \((\gamma, \alpha) - P_S\)-irresolute.

**Proof.** It is clear. \( \square \)

**Theorem 5.33.** Let \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \rho) \) be functions. Then the composition function \( g \circ f: (X, \tau) \to (Z, \rho) \) is \( \gamma - P_S\)-continuous if \( f \) and \( g \) satisfy one of the following conditions:

1. \( f \) is \((\gamma, \beta) - P_S\)-irresolute and \( g \) is \( \beta - P_S\)-continuous.

2. \( f \) is \((\gamma, \beta) - P_S\)-continuous and \( g \) is \( \beta\)-continuous.

**Proof.** The proof is similar to Theorem 5.31. \( \square \)

**Proposition 5.34.** Let \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \rho) \) be any functions and \( \alpha \) be an operation on \( \rho \). Then the following are holds:

1. If \( f \) is \( \beta \)-open (\( \beta \)-closed) and \( g \) is \((\beta, \alpha) - P_S\)-open (\((\beta, \alpha) - P_S\)-closed), then \( g \circ f: (X, \tau) \to (Z, \rho) \) is \( \alpha - P_S\)-open (\( \alpha - P_S\)-closed).
2. If $f$ is $(\gamma, \beta)$-$P_S$-open ($(\gamma, \beta)$-$P_S$-closed) and $g$ is $(\beta, \alpha)$-$P_S$-open ($(\beta, \alpha)$-$P_S$-closed), then $g \circ f : (X, \tau) \rightarrow (Z, \rho)$ is $(\gamma, \alpha)$-$P_S$-open ($(\gamma, \alpha)$-$P_S$-closed).

3. If $f$ is $(\gamma, \beta P_S)$-open ($(\gamma, \beta P_S)$-closed) and $g$ is $(\beta, \alpha)$-$P_S$-open ($(\beta, \alpha)$-$P_S$-closed), then $g \circ f : (X, \tau) \rightarrow (Z, \rho)$ is $(\gamma, \alpha P_S)$-open ($(\gamma, \alpha P_S)$-closed).

Proof. It is clear. □

Proposition 5.35. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a function, $g : (Y, \sigma) \rightarrow (Z, \rho)$ is $(\beta, \alpha)$-$P_S$-open and injective, and $g \circ f : (X, \tau) \rightarrow (Z, \rho)$ is $(\gamma, \alpha)$-$P_S$-irresolute. Then $f$ is $(\gamma, \beta)$-$P_S$-irresolute.

Proof. Let $V$ be a $\beta$-$P_S$-open subset of $Y$. Since $g$ is $(\beta, \alpha)$-$P_S$-open, $g(V)$ is $\alpha$-$P_S$-open subset of $Z$. Since $g \circ f$ is $(\gamma, \alpha)$-$P_S$-irresolute and $g$ is injective, then $f^{-1}(V) = f^{-1}(g(V)) = (g \circ f)(g(V))$ is $\gamma$-$P_S$-open in $X$, which proves that $f$ is $(\gamma, \beta)$-$P_S$-irresolute. □

Proposition 5.36. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \rho)$ be any functions and $\alpha$ be an operation on $\rho$. Then the following are holds:

1. If $f$ is $(\gamma, \beta)$-$P_S$-open and surjective, and $g \circ f : (X, \tau) \rightarrow (Z, \rho)$ is $(\gamma, \alpha)$-$P_S$-irresolute, then $g$ is $(\gamma, \beta)$-$P_S$-irresolute.

2. If $f$ is $(\gamma, \beta)$-$P_S$-irresolute and surjective, and $g \circ f : (X, \tau) \rightarrow (Z, \rho)$ is $(\gamma, \alpha)$-$P_S$-open, then $g$ is $(\beta, \alpha)$-$P_S$-open.

Proof. Similar to Proposition 5.35. □

References


