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(γ, β)-P_S-IRRESOLUTE AND (γ, β)-P_S-CONTINUOUS FUNCTIONS

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Abstract: This paper introduces some new types of functions called (γ, β)-P_S-irresolute and (γ, β)-P_S-continuous by using γ-P_S-open sets in topological spaces (X, τ). From γ-P_S-open and γ-P_S-closed sets, some other types of γ-P_S-functions can also be defined. Moreover, some basic properties and preservation theorems of these functions are obtained. In addition, we investigate basic characterizations and properties of these γ-P_S- functions. Finally, some compositions of these γ-P_S- functions are given.

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1. Introduction

Kasahara [7] defined the concept of α-closed graphs of an operation on τ. Later, Ogata [10] renamed the operation α as γ operation on τ. He defined γ-open sets and introduced the notion of τ_γ which is the class of all γ-open sets in a topological space (X, τ). Further study by Krishnan and Balachandran ([8], [9]) defined two types of sets called γ-preopen and γ-semiopen sets. Recently, Asaad, Ahmad and Omar [1] introduced the notion of γ-regular-open sets. They

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also introduced the notion of γ-P_S-open sets [2] which lies strictly between the classes of γ-regular-open set and γ-preopen set. By using this set, they defined a new type of function called γ-P_S-continuous and studies some of its basic properties [3].

In the present paper, we define some new types of γ-P_S- functions called (γ,β)-P_S-irresolute and (γ,β)-P_S-continuous by using γ-P_S-open sets in topological spaces (X,τ). In addition, we give some basic characterizations and properties of these γ-P_S- functions by using γ-P_S-open and γ-P_S-closed sets are introduced. Finally, some compositions of these γ-P_S- functions are given.

2. Preliminaries

Throughout this paper, spaces (X,τ) and (Y,σ) (or simply X and Y) will always means topological spaces on which no separation axioms are assumed unless explicitly stated. An operation γ on the topology τ on X is a mapping $\gamma\colon \tau \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U [10]. A nonempty subset A of a space (X,τ) with an operation γ on τ is said to be γ-open if for each $x \in A$, there exists an open set U such that $x \subseteq U$ and $\gamma(U) \subseteq A$ [10]. The complement of a γ-open set is called a γ-closed. The τ_γ-closure of a subset A of X with an operation γ on τ is defined as the intersection of all γ-closed sets of X containing A and it is denoted by τ_γ-$Cl(A)$ [10], and the τ_γ-interior of a subset A of X with an operation γ on τ is defined as the union of all γ-open sets of X contained in A and it is denoted by τ_γ-$Int(A)$ [9]. A topological space (X,τ) is said to be γ-regular if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U) \subseteq V$ [10]. Throughout of this paper, γ and β be operations on τ and σ respectively.

Now we begin to recall some known notions which are useful in the sequel.

Definition 2.1. A subset A of a topological space (X,τ) is said to be:

1. γ-regular-open if $A = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))$ and γ-regular-closed if $A = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))$ [1].

2. γ-preopen if $A \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))$ and γ-preclosed if τ_γ-$Cl(\tau_\gamma$-$Int(A)) \subseteq A$ [8].

3. γ-semiopen if $A \subseteq \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))$ and γ-semiclosed if τ_γ-$Int(\tau_\gamma$-$Cl(A)) \subseteq A$ [9].

4. γ-dense if τ_γ-$Cl(A) = X$ [6].

Definition 2.2. [2] A γ-preopen subset A of a topological space (X,τ) is called γ-P_S-open if for each $x \in A$, there exists a γ-semiclosed set F such that $x \in F \subseteq A$. The complement of a γ-P_S-open set of X is called γ-P_S-closed.

The class of all γ-P_S-open and γ-preopen subsets of a topological space (X,τ) are denoted by τ_γ-$P_SO(X)$ and τ_γ-$PO(X)$ respectively.

Lemma 2.3. *[2] A subset A of X is γ-P_S-open if and only if A is γ-preopen set and it is a union of γ-semiclosed sets.*

Definition 2.4. A subset N of a topological space (X,τ) is called a γ-P_S-neighbourhood of a point $x \in X$, if there exists a γ-P_S-open set U in X containinig x such that $U \subseteq N$.

Definition 2.5. [2] For any subset A of a space X. Then:

1. the γ-P_S-boundary of A is defined as τ_γ-$P_SCl(A) \setminus \tau_\gamma$-$P_SInt(A)$ and it is denoted by τ_γ-$P_SBd(A)$.

2. the γ-P_S-derived set of A is defined as $\{x :$ for every γ-P_S-open set U containing x, $U \cap A \setminus \{x\} \neq \phi\}$ and it is denoted by τ_γ-$P_SD(A)$.

Lemma 2.6. *[2] For any subset A of a space X. Then the following statements are true:*

1. *τ_γ-$P_SCl(A)$ is the smallest γ-P_S-closed set of X containing A.*

2. *τ_γ-$P_SInt(A)$ is the largest γ-P_S-open set of X contained in A.*

3. *A is γ-P_S-closed if and only if τ_γ-$P_SCl(A) = A$, and A is γ-P_S-open if and only if τ_γ-$P_SInt(A) = A$.*

4. *τ_γ-$P_SCl(A) = X \setminus \tau_\gamma$-$P_SInt(X \setminus A)$ and τ_γ-$P_SInt(A) = X \setminus \tau_\gamma$-$P_SCl(X \setminus A)$.*

5. *A is γ-P_S-closed if and only if τ_γ-$P_SBd(A) \subseteq A$.*

6. *τ_γ-$P_SD(A) \subseteq \tau_\gamma$-$P_SCl(A)$.*

7. *A is γ-P_S-closed if and only if τ_γ-$P_SD(A) \subseteq A$.*

Definition 2.7. [4] A subset A of a space (X,τ) is said to be γ-P_S-generalized closed (γ-P_S-g-closed) if τ_γ-$P_SCl(A) \subseteq G$ whenever $A \subseteq G$ and G is a γ-P_S-open set in X.

Definition 2.8. A topological space (X,τ) is said to be:

1. γ-locally indiscrete if every γ-open subset of X is γ-closed, or every γ-closed subset of X is γ-open [1].

2. γ-hyperconnected if every nonempty γ-open subset of X is γ-dense [1].

3. γ-P_S-$T_{\frac{1}{2}}$ if every γ-P_S-g-closed set of X is γ-P_S-closed [4].

4. γ-semiT_1 if for each pair of distinct points x, y in X, there exist two γ-semiopen sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$ [9].

Theorem 2.9. *The following statements are true for any space (X, τ):*

1. *If X is γ-locally indiscrete, then τ_γ-$P_SO(X) = \tau_\gamma$ [2].*

2. *If X is γ-semiT_1, then τ_γ-$P_SO(X) = \tau_\gamma$-$PO(X)$ [2].*

3. *If X is γ-hyperconnected if and only if τ_γ-$P_SO(X) = \{\phi, X\}$ [2].*

4. *X is γ-P_S-$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is γ-P_S-closed or γ-P_S-open [4].*

5. *X is γ-regular, then $\tau_\gamma = \tau$ [10].*

Definition 2.10. Let (X, τ) and (Y, σ) be two topological spaces. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called:

1. γ-P_S-continuous if for each P_S-open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$ [3].

2. (γ, β)-precontinuous if for each β-preopen set V of Y containing $f(x)$, there exists a γ-preopen set U of X containing x such that $f(U) \subseteq V$ [8].

3. γ-continuous if for each open set V of Y containing $f(x)$, there exists a γ-open set U of X containing x such that $f(U) \subseteq V$ [5].

4. β-P_S-open (resp., β-open and β-P_S-closed) if for every open (resp., open and closed) set V of X, $f(V)$ is β-P_S-open (resp., β-open and β-P_S-closed) set in Y [3].

3. (γ, β)-P_S-Irresolute and (γ, β)-P_S-Continuous Functions

In this section, we introduce three types of γ-P_S- functions called (γ, β)-P_S-irresolute and (γ, β)-P_S-continuous by using γ-P_S-open set. Also we give relations between these functions and γ-P_S-continuous function.

Definition 3.1. Let (X, τ) and (Y, σ) be two topological spaces. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called (γ, β)-P_S-irresolute (resp., (γ, β)-P_S-continuous) at a point $x \in X$ if for each β-P_S-open (resp., β-open) set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. If f is (γ, β)-P_S-irresolute (resp., (γ, β)-P_S-continuous) at every point x in X, then f is said to be (γ, β)-P_S-irresolute (resp., (γ, β)-P_S-continuous).

Remark 3.2. It is clear from the Definition 2.10 (1) and Definition 3.1 that every γ-P_S-continuous function is (γ, β)-P_S-continuous since every β-open set is open, where β is an operation on σ. However, the converse is not true in general as it can be seen from the following example.

Example 3.3. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\sigma = \{\phi, Y, \{2\}, \{3\}, \{2, 3\}\}$. Define operations $\gamma\colon \tau \to P(X)$ and $\beta\colon \sigma \to P(Y)$ as follows: for every $A \in \tau$ and $B \in \sigma$

$$\gamma(A) = \begin{cases} A & \text{if } a \in A \\ Cl(A) & \text{if } a \notin A \end{cases}$$

$$\beta(B) = \begin{cases} B & \text{if } B = \{2\} \\ Cl(B) & \text{if } B \neq \{2\} \end{cases}$$

Then $\sigma_\beta = \{\phi, \{2\}, Y\}$.
Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 2 & \text{if } x = a \\ 3 & \text{if } x = b \\ 1 & \text{if } x = c \end{cases}$$

Clearly, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}$, τ_γ-$P_SO(X) = \{\phi, \{a\}, \{a, c\}, \{b, c\}, X\}$ and $\sigma_\beta = \{\phi, \{2\}, Y\}$. Let $f\colon (X, \tau) \to (X, \sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 2 & \text{if } x = a \\ 3 & \text{if } x = b \\ 1 & \text{if } x = c \end{cases}$$

Then f is (γ, β)-P_S-continuous, but it is not γ-P_S-continuous since $\{3\}$ is an open set in (Y, σ) containing $f(b) = 3$, but there exist no γ-P_S-open set U in (X, τ) containing b such that $f(U) \subseteq \{3\}$.

Remark 3.4. The relation between (γ, β)-P_S-irresolute function and (γ, β)-P_S-continuous function are independent. Similarly the relation between (γ, β)-P_S-irresolute function and γ-P_S-continuous function are independent, as shown from the following examples.

Example 3.5. Let (X, τ) be a topological space as in Example 3.3. Suppose that $Y = \{1, 2, 3\}$ and $\sigma = \{\phi, Y, \{2\}, \{2, 3\}\}$ be a topology on Y. Define an operation β on σ such that $\beta\colon \sigma \to P(Y)$ by $\beta(B) = B$ for all $B \in \sigma$. Then σ_β-$P_SO(Y) = \{\phi, Y\}$.
Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 3 & \text{if } x = a \\ 2 & \text{if } x = b \\ 1 & \text{if } x = c \end{cases}$$

Then the function f is (γ, β)-P_S-irresolute, but f is not (γ, β)-P_S-continuous since $\{2\}$ is a β-open set in (Y, σ) containing $f(b) = 2$, but there exist no γ-P_S-open set U in (X, τ) containing b such that $f(U) \subseteq \{2\}$. By Remark 3.2, f is not γ-P_S-continuous.

Example 3.6. Consider the space $X = \{a, b, c\}$ with the topologies $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Define the operations γ and β on τ and σ respectively as follows: For every $A \in \tau$, $\gamma(A) = A$ and for every $B \in \sigma$

$$\beta(B) = \begin{cases} B & \text{if } c \in B \\ Cl(B) & \text{if } c \notin B \end{cases}$$

Obviously, $\tau_\gamma = \tau = \tau_\gamma$-$P_SO(X)$, $\sigma_\beta = \{\phi, X, \{c\}, \{a, b\}, \{b, c\}\}$ and σ_β-$P_SO(X) = \{\phi, X, \{c\}, \{a, b\}, \{a, c\}\}$.
Define a function $f\colon (X, \tau) \to (X, \sigma)$ as follows:

$$f(x) = \begin{cases} b & \text{if } x \in \{a, c\} \\ a & \text{if } x = b \end{cases}$$

So, the function f is both (γ, β)-P_S-continuous and γ-P_S-continuous, but f is not (γ, β)-P_S-irresolute since $\{a, c\}$ is a β-P_S-open set in (X, σ) containing $f(b) = a$, there exist no γ-P_S-open set U in (X, τ) containing c such that $f(U) \subseteq \{a, c\}$.

4. Characterizations

We start with the most important characterizations of (γ, β)-P_S-irresolute functions.

Theorem 4.1. *For any function $f\colon (X, \tau) \to (Y, \sigma)$. The following properties of f are equivalent:*

1. *f is (γ, β)-P_S-irresolute.*
2. *The inverse image of every β-P_S-open set of Y is γ-P_S-open set in X.*
3. *The inverse image of every β-P_S-closed set of Y is γ-P_S-closed set in X.*
4. *$f(\tau_\gamma\text{-}P_SCl(A)) \subseteq \sigma_\beta\text{-}P_SCl(f(A))$, for every subset A of X.*
5. *$\tau_\gamma\text{-}P_SCl(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}P_SCl(B))$, for every subset B of Y.*
6. *$f^{-1}(\sigma_\beta\text{-}P_SInt(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$, for every subset B of Y.*
7. *$\sigma_\beta\text{-}P_SInt(f(A)) \subseteq f(\tau_\gamma\text{-}P_SInt(A))$, for every subset A of X.*

Proof. (1) $\Rightarrow$ (2) Let V be any β-P_S-open set in Y. We have to show that $f^{-1}(V)$ is γ-P_S-open set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By (1), there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. Which implies that $x \in U \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is γ-P_S-open set in X.

(2) $\Rightarrow$ (3) Let F be any β-P_S-closed set of Y. Then $Y \backslash F$ is a β-P_S-open set of Y. By (2), $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$ is γ-P_S-open set in X and hence $f^{-1}(F)$ is γ-P_S-closed set in X.

(3) $\Rightarrow$ (4) Let A be any subset of X. Then $f(A) \subseteq \sigma_\beta\text{-}P_SCl(f(A))$ and hence $A \subseteq f^{-1}(\sigma_\beta\text{-}P_SCl(f(A)))$. Since $\sigma_\beta\text{-}P_SCl(f(A))$ is β-P_S-closed set in Y. Then by (3), we have $f^{-1}(\sigma_\beta\text{-}P_SCl(f(A)))$ is γ-P_S-closed set in X. Therefore, $\tau_\gamma\text{-}P_SCl(A) \subseteq f^{-1}(\sigma_\beta\text{-}P_SCl(f(A)))$. Hence $f(\tau_\gamma\text{-}P_SCl(A)) \subseteq \sigma_\beta\text{-}P_SCl(f(A))$.

(4) $\Rightarrow$ (5) Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (4), we have $f(\tau_\gamma\text{-}P_SCl(f^{-1}(B))) \subseteq \sigma_\beta\text{-}P_SCl(f(f^{-1}(B))) = \sigma_\beta\text{-}P_SCl(B)$. Hence $\tau_\gamma\text{-}P_SCl(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}P_SCl(B))$.

(5) $\Leftrightarrow$ (6) Let B be any subset of Y. Then apply (5) to $Y \backslash B$ we obtain $\tau_\gamma\text{-}P_SCl(f^{-1}(Y \backslash B)) \subseteq f^{-1}(\sigma_\beta\text{-}P_SCl(Y \backslash B)) \Leftrightarrow \tau_\gamma\text{-}P_SCl(X \backslash f^{-1}(B)) \subseteq f^{-1}(Y \backslash \sigma_\beta\text{-}P_SInt(B)) \Leftrightarrow X \backslash \tau_\gamma\text{-}P_SInt(f^{-1}(B)) \subseteq X \backslash f^{-1}(\sigma_\beta\text{-}P_SInt(B)) \Leftrightarrow f^{-1}(\sigma_\beta\text{-}P_SInt(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$. Therefore, $f^{-1}(\sigma_\beta\text{-}P_SInt(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$.

(6) $\Rightarrow$ (7) Let A be any subset of X. Then $f(A)$ is a subset of Y. By (6), we have $f^{-1}(\sigma_\beta\text{-}P_SInt(f(A))) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(f(A))) = \tau_\gamma\text{-}P_SInt(A)$. Therefore, $\sigma_\beta\text{-}P_SInt(f(A)) \subseteq f(\tau_\gamma\text{-}P_SInt(A))$.

(7) $\Rightarrow$ (1) Let $x \in X$ and let V be any β-P_S-open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. By (7), we have $\sigma_\beta\text{-}P_SInt(f(f^{-1}(V))) \subseteq f(\tau_\gamma\text{-}P_SInt(f^{-1}(V)))$. Then $\sigma_\beta\text{-}P_SInt(V) \subseteq f(\tau_\gamma\text{-}P_SInt(f^{-1}(V)))$. Since V is a β-P_S-open set. Then $V \subseteq f(\tau_\gamma\text{-}P_SInt(f^{-1}(V)))$ implies that $f^{-1}(V) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is γ-P_S-open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is (γ,β)-P_S-irresolute function. □

Theorem 4.2. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function. Then the following statements are equivalent:*

1. *f is (γ,β)-P_S-continuous.*
2. *$f^{-1}(V)$ is γ-P_S-open set in X, for every β-open set V in Y.*
3. *$f^{-1}(F)$ is γ-P_S-closed set in X, for every β-closed set F in Y.*
4. *$f(\tau_\gamma\text{-}P_SCl(A)) \subseteq \sigma_\beta\text{-}Cl(f(A))$, for every subset A of X.*
5. *$\sigma_\beta\text{-}Int(f(A)) \subseteq f(\tau_\gamma\text{-}P_SInt(A))$, for every subset A of X.*
6. *$f^{-1}(\sigma_\beta\text{-}Int(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$, for every subset B of Y.*
7. *$\tau_\gamma\text{-}P_SCl(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}Cl(B))$, for every subset B of Y.*

Proof. Similar to Theorem 4.1 and hence it is ommited. □

Theorem 4.3. *The following properties are equivalent for any function $f\colon (X,\tau) \to (Y,\sigma)$:*

1. *f is (γ,β)-P_S-irresolute.*
2. *For every $x \in X$ and for every β-P_S-neighbourhood N of Y such that $f(x) \in N$, there exists a γ-P_S-neighbourhood M of X such that $x \in M$ and $f(M) \subseteq N$.*
3. *The inverse image of every β-P_S-neighbourhood of $f(x)$ is γ-P_S- neighbourhood of $x \in X$.*

Proof. It is clear and hence it is omitted. □

Lemma 4.4. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be a γ-continuous and β-open function, then the following statements are true:*

1. *If V is β-preopen set of Y, then $f^{-1}(V)$ is γ-preopen set in X.*
2. *If F is β-semiclosed set of Y, then $f^{-1}(F)$ is γ-semiclosed set in X.*

Lemma 4.5. *If $f\colon (X, \tau) \to (Y, \sigma)$ is a γ-continuous and β-open function and V is β-P_S-open set of Y, then $f^{-1}(V)$ is γ-P_S-open set in X.*

Proof. Let V be a β-P_S-open set of Y, then V is a β-preopen set of Y and $V = \bigcup_{i\in I} F_i$ where F_i is β-semiclosed set in Y for each i. Then $f^{-1}(V) = f^{-1}(\bigcup_{i\in I} F_i) = \bigcup_{i\in I} f^{-1}(F_i)$ where F_i is β-semiclosed set in Y for each i. Since f is a γ-continuous and β-open function. Then by Lemma 4.4 (1), $f^{-1}(V)$ is γ-preopen set of X and by Lemma 4.4 (2), $f^{-1}(F_i)$ is γ-semiclosed set of X for each i. Hence by Lemma 2.3, $f^{-1}(V)$ is γ-P_S-open set in X. □

Corollary 4.6. *If $f\colon (X, \tau) \to (Y, \sigma)$ is a γ-continuous and β-open function and F is β-P_S-closed set of Y, then $f^{-1}(F)$ is γ-P_S-closed set in X.*

Lemma 4.7. *If a function $f\colon (X, \tau) \to (Y, \sigma)$ is both γ-continuous and β-open, then f is (γ, β)-P_S-irresolute.*

Proof. The proof follows directly from Lemma 4.5 and Theorem 4.1. □

Some other characterizations of (γ, β)-P_S-irresolute functions are mentioned in the following.

Theorem 4.8. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function. Then the following properties are equivalent:*

1. *f is (γ, β)-P_S-irresolute.*
2. *τ_γ-$P_SBd(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SBd(B))$, for each subset B of Y.*
3. *$f(\tau_\gamma$-$P_SBd(A)) \subseteq \sigma_\beta$-$P_SBd(f(A))$, for each subset A of X.*

Proof. (1) ⇒ (2). Let f be a (γ, β)-P_S-irresolute function and B be any subset of (Y, σ). Then by Theorem 4.1 (2) and (5), we have τ_γ-$P_SBd(f^{-1}(B)) = \tau_\gamma$-$P_SCl(f^{-1}(B))\backslash\tau_\gamma$-$P_SInt(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SCl(B))\backslash\tau_\gamma$-$P_SInt(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SCl(B))\backslash\tau_\gamma$-$P_SInt(f^{-1}(\sigma_\beta$-$P_SInt(B))) = f^{-1}(\sigma_\beta$-$P_SCl(B))\backslash f^{-1}(\sigma_\beta$-$P_SInt(B)) = f^{-1}(\sigma_\beta$-$P_SCl(B)\backslash\sigma_\beta$-$P_SInt(B)) = f^{-1}(\sigma_\beta$-$P_SBd(B))$. Therefore, τ_γ-$P_SBd(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SBd(B))$.

(2) ⇒ (3). Let A be any subset of X. Then $f(A)$ is a subset of Y. Then by (2), we have τ_γ-$P_SBd(f^{-1}(f(A))) \subseteq f^{-1}(\sigma_\beta$-$P_SBd(f(A)))$ implies

that τ_γ-$P_SBd(A) \subseteq f^{-1}(\sigma_\beta$-$P_SBd(f(A)))$ and hence $f(\tau_\gamma$-$P_SBd(A)) \subseteq \sigma_\beta$-$P_SBd(f(A))$. This completes the proof.

(3) $\Rightarrow$ (1). Let E be any β-P_S-closed set in Y. Then $f^{-1}(E)$ is a subset of X. So by using part (3), we have $f(\tau_\gamma$-$P_SBd(f^{-1}(E))) \subseteq \sigma_\beta$-$P_SBd(f(f^{-1}(E))) = \sigma_\beta$-$P_SBd(E)$ implies that $f(\tau_\gamma$-$P_SBd(f^{-1}(E))) \subseteq \sigma_\beta$-$P_SBd(E) \subseteq \sigma_\beta$-$P_SCl(E) = E$ and hence $f(\tau_\gamma$-$P_SBd(f^{-1}(E))) \subseteq E$. This implies that τ_γ-$P_SBd(f^{-1}(E)) \subseteq f^{-1}(E)$. Thus, by Lemma 2.6 (5), $f^{-1}(E)$ is γ-P_S-closed set in X. Consequently by Theorem 4.1, f is (γ, β)-P_S-irresolute function. □

Theorem 4.9. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function. Then the following properties are equivalent:*

1. *f is (γ, β)-P_S-continuous.*
2. *For each subset A in X, $f(\tau_\gamma$-$P_SBd(A)) \subseteq \sigma_\beta$-$Bd(f(A))$.*
3. *For each subset B in Y, τ_γ-$P_SBd(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$Bd(B))$.*

Proof. The proof is similar to Theorem 4.8, and hence it is omitted. □

Theorem 4.10. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function. Then the following properties are equivalent:*

1. *f is (γ, β)-P_S-irresolute.*
2. *$f(\tau_\gamma$-$P_SD(A)) \subseteq \sigma_\beta$-$P_SCl(f(A))$, for each subset A of X.*
3. *τ_γ-$P_SD(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SCl(B))$, for each subset B of Y.*

Proof. (1) $\Rightarrow$ (2). Let f be a (γ, β)-P_S-continuous function and A be any subset of X. Then by Theorem 4.1 (4), we have $f(\tau_\gamma$-$P_SCl(A)) \subseteq \sigma_\beta$-$P_SCl(f(A))$. Then by Lemma 2.6 (6), we obtain $f(\tau_\gamma$-$P_SD(A)) \subseteq f(\tau_\gamma$-$P_SCl(A))$ which implies that $f(\tau_\gamma$-$P_SD(A)) \subseteq \sigma_\beta$-$P_SCl(f(A))$.

(2) $\Rightarrow$ (3). Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. Then by hypothesis, we get $f(\tau_\gamma$-$P_SD(f^{-1}(B))) \subseteq \sigma_\beta$-$P_SCl(f(f^{-1}(B))) = \sigma_\beta$-$P_SCl(B)$ and hence τ_γ-$P_SD(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SCl(B))$. This completes the proof.

(3) $\Rightarrow$ (1). Let F be any β-P_S-closed set in Y. Then by (3), we have τ_γ-$P_SD(f^{-1}(F)) \subseteq f^{-1}(\sigma_\beta$-$P_SCl(F)) = f^{-1}(F)$ and hence τ_γ-$P_SD(f^{-1}(F)) \subseteq f^{-1}(F)$. So by Lemma 2.6 (7), we get $f^{-1}(F)$ is γ-P_S-closed set in X. Therefore, by Theorem 4.1, f is (γ, β)-P_S-irresolute function. □

Theorem 4.11. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function. Then the following properties are equivalent:*

1. *f is (γ, β)-P_S-continuous.*

2. *$f(\tau_\gamma\text{-}P_S D(A)) \subseteq \sigma_\beta\text{-}Cl(f(A))$, for each subset A of X.*

3. *$\tau_\gamma\text{-}P_S D(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta\text{-}Cl(B))$, for each subset B of Y.*

Proof. The proof is similar to Theorem 4.10, and hence it is omitted. □

Proposition 4.12. *If a function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-irresolute, then for each $x \in X$ and each β-P_S-open set V of Y containing $f(x)$, there exists a γ-semiclosed set F in X such that $x \in F$ and $f(F) \subseteq V$. Furthermore, if f is (γ, β)-precontinuous, then the converse also holds.*

Proof. Suppose f be a (γ, β)-P_S-irresolute function and let V be any β-P_S-open set of Y such that $f(x) \in V$, for each $x \in X$. Then there exists a γ-P_S-open set U of X such that $x \in U$ and $f(U) \subseteq V$. Since U is γ-P_S-open set. Then for each $x \in U$, there exists a γ-semiclosed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$.

Now suppose that f is (γ, β)-precontinuous function. Let V be any β-P_S-open set of Y. We have to show that $f^{-1}(V)$ is γ-P_S-open set in X. Since every β-P_S-open set is β-preopen, then $f^{-1}(V)$ is γ-preopen set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, there exists a γ-semiclosed set F of X containing x such that $f(F) \subseteq V$. Which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, by Definition 2.2, $f^{-1}(V)$ is γ-P_S-open set in X. Hence by Theorem 4.1, f is (γ, β)-P_S-irresolute. This completes the proof. □

5. Properties

Theorem 5.1. *Let (X, τ) be γ-semiT_1 space and (Y, σ) be β-semiT_1 space. A function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-irresolute if and only if f is (γ, β)-precontinuous.*

Proof. This is an immediate consequence of Theorem 2.9 (2). □

Theorem 5.2. *Let (Y, σ) be β-locally indiscrete space. A function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-irresolute if and only if f is (γ, β)-P_S-continuous.*

Proof. Follows directly from Theorem 2.9 (1). □

Theorem 5.3. *Let (Y, σ) be β-regular space. A function $f\colon (X, \tau) \to (Y, \sigma)$ is γ-P_S-continuous if and only if f is (γ, β)-P_S-continuous.*

Proof. Follows directly from Theorem 2.9 (5). □

Theorem 5.4. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function. If (Y,σ) is β-hyperconnected space, then f is (γ,β)-P_S-irresolute.*

Proof. This is an immediate consequence of Theorem 2.9 (3). □

Proposition 5.5. *If a function $f\colon (X,\tau) \to (Y,\sigma)$ is (γ,β)-P_S-irresolute (resp. (γ,β)-P_S-continuous), then the following properties are true:*

1. *for each $x \in X$ and each β-P_S-open (resp. β-open) set V of Y such that $f(x) \in V$, there exists a γ-preopen set U of X such that $x \in U$ and $f(U) \subseteq V$.*

2. *for each $x \in X$ and each β-regular-open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$.*

Proof. 1) Since every γ-P_S-open set of X is γ-preopen, then by using this in Definition 3.1 we get the proof.

2) Since every β-regular-open set of Y is both β-open and β-P_S-open, then the proof follows directly from Definition 3.1. □

Proposition 5.6. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is (γ,β)-P_S-irresolute (resp. (γ,β)-P_S-continuous), if the following properties are true:*

1. *for each $x \in X$ and each β-preopen set V of Y such that $f(x) \in V$, there exists a γ-P_S-open set U of X such that $x \in U$ and $f(U) \subseteq V$.*

2. *for each $x \in X$ and each β-P_S-open (resp. β-open) set V of Y containing $f(x)$, there exists a γ-regular-open set U of X containing x such that $f(U) \subseteq V$.*

Proof. 1) The proof is clear since every β-P_S-open (resp. β-open) set of Y is β-preopen.

2) Obvious since every γ-regular-open set of X is γ-P_S-open and hence it is omitted. □

Theorem 5.7. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function, then: $X \backslash \tau_\gamma$-$P_SC(f) = \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is a β-P_S-open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$, where τ_γ-$P_SC(f)$ denotes the set of points at which f is (γ,β)-P_S-irresolute function.*

Proof. Let $x \in \tau_\gamma$-$P_SC(f)$. Then there exists β-P_S-open set V in (Y,σ) containing $f(x)$ such that $f(U) \nsubseteq V$ for every γ-P_S-open set U of (X,τ) containing x. Hence $U \cap X\backslash f^{-1}(V) \neq \phi$ for every γ-P_S-open set U of (X,τ) containing x. Therefore, $x \in \tau_\gamma$-$P_SCl(X\backslash f^{-1}(V))$. Then $x \in f^{-1}(V) \cap \tau_\gamma$-$P_SCl(X\backslash f^{-1}(V)) \subseteq \tau_\gamma$-$P_SCl(f^{-1}(V)) \cap \tau_\gamma$-$P_SCl(X\backslash f^{-1}(V)) = \tau_\gamma$-$P_SBd(f^{-1}(V))$. Then $X\backslash\tau_\gamma$-$P_SC(f) \subseteq \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is β-P_S-open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$.

Conversely, let $x \notin X\backslash\tau_\gamma$-$P_SC(f)$. Then for each β-P_S-open set V in (Y,σ) containing $f(x)$, $f^{-1}(V)$ is γ-P_S-open set of (X,τ) containing x. So $x \in \tau_\gamma$-$P_SInt(f^{-1}(V))$ and hence $x \notin \tau_\gamma$-$P_SBd(f^{-1}(V))$ for every β-P_S-open set V in (Y,σ) containing $f(x)$. Therefore, $X\backslash\tau_\gamma$-$P_SC(f) \supseteq \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is β-P_S-open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$. □

The proof of the following theorem is similar to Theorem 5.7 and is thus omitted.

Theorem 5.8. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function, then: $X\backslash\tau_\gamma$-$P_SC(f) = \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is a β-open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$, where τ_γ-$P_SC(f)$ denotes the set of points at which f is (γ,β)-P_S-continuous function.*

Now, we will define more types of γ-P_S- functions by using γ-P_S-open set which are defined as follows.

Definition 5.9. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called (γ,β)-P_S-open (resp. $(\gamma,\beta P_S)$-open) if for every γ-P_S-open (resp. γ-open) set V of X, $f(V)$ is β-P_S-open set in Y.

Definition 5.10. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called (γ,β)-P_S-closed (resp. $(\gamma,\beta P_S)$-open) if for every γ-P_S-closed (γ-closed) set F of X, $f(F)$ is β-P_S-closed set in Y.

Theorem 5.11. *A function $f\colon (X,\tau) \to (Y,\sigma)$ is (γ,β)-P_S-open if and only if for every $x \in X$ and for every γ-P_S-neighbourhood N of x, there exists a β-P_S-neighbourhood M of Y such that $f(x) \in M$ and $M \subseteq f(N)$.*

Proof. Obvious. □

Theorem 5.12. *The following statements are equivalent for a function $f\colon (X,\tau) \to (Y,\sigma)$:*

1. *f is (γ,β)-P_S-open.*

2. *$f(\tau_\gamma$-$P_SInt(A)) \subseteq \sigma_\beta$-$P_SInt(f(A))$, for every $A \subseteq X$.*

3. *τ_γ-$P_SInt(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SInt(B))$, for every $B \subseteq Y$.*

Proof. The proof is similar to Theorem 4.1. □

Theorem 5.13. *The following properties of f are equivalent for a function $f\colon (X,\tau) \to (Y,\sigma)$:*

1. *f is (γ,β)-P_S-closed.*

2. *$f^{-1}(\sigma_\beta$-$P_SCl(B)) \subseteq \tau_\gamma$-$P_SCl(f^{-1}(B))$, for every $B \subseteq Y$.*

3. *σ_β-$P_SCl(f(A)) \subseteq f(\tau_\gamma$-$P_SCl(A))$, for every $A \subseteq X$.*

4. *σ_β-$P_SD(f(A)) \subseteq f(\tau_\gamma$-$P_SCl(A))$, for every $A \subseteq X$.*

Proof. The proof is similar to Theorem 4.1. □

Definition 5.14. Let $id\colon \tau \to P(X)$ be the identity operation. If f is (id,β)-P_S-closed, then for every γ-P_S-closed set F of X, $f(F)$ is β-P_S-closed set in Y.

Theorem 5.15. *If a function f is bijective and $f^{-1}\colon (Y,\sigma) \to (X,\tau)$ is (id,β)-P_S-irresolute, then f is (id,β)-P_S-closed.*

Proof. Follows from Definition 5.14 and Definition 3.1. □

Theorem 5.16. *Suppose that a function $f\colon (X,\tau) \to (Y,\sigma)$ is both (γ,β)-P_S-irresolute and (γ,β)-P_S-closed, then:*

1. *For every γ-P_S-g-closed set A of (X,τ), the image $f(A)$ is β-P_S-g-closed in (Y,σ).*

2. *For every β-P_S-g-closed set B of (Y,σ) the inverse set $f^{-1}(B)$ is γ-P_S-g-closed in (X,τ).*

Proof. (1) Let G be any β-P_S-open set in (Y,σ) such that $f(A) \subseteq G$. Since f is (γ,β)-P_S-irresolute function, then by Theorem 4.1 (2), $f^{-1}(G)$ is γ-P_S-open set in (X,τ). Since A is γ-P_S-g-closed and $A \subseteq f^{-1}(G)$, we have τ_γ-$P_SCl(A) \subseteq f^{-1}(G)$, and hence $f(\tau_\gamma$-$P_SCl(A)) \subseteq G$. Since τ_γ-$P_SCl(A)$ is γ-P_S-closed set and f is (γ,β)-P_S-closed, then $f(\tau_\gamma$-$P_SCl(A))$ is β-P_S-closed set in Y. Therefore, σ_β-$P_SCl(f(A)) \subseteq \sigma_\beta$-$P_SCl(f(\tau_\gamma$-$P_SCl(A))) = f(\tau_\gamma$-$P_SCl(A)) \subseteq G$. This implies that $f(A)$ is β-P_S-g-closed in (Y,σ).

(2) Let H be any γ-P_S-open set of (X,τ) such that $f^{-1}(B) \subseteq H$. Let $C = \tau_\gamma$-$P_SCl(f^{-1}(B)) \cap (X \backslash H)$, then C is γ-P_S-closed set in (X,τ). Since f is

(γ, β)-P_S-closed function. Then $f(C)$ is β-P_S-closed in (Y, σ). Since f is (γ, β)-P_S-irresolute function, then by using Theorem 4.1 (4), we have $f(C) = f(\tau_\gamma$-$P_S Cl(f^{-1}(B))) \cap f(X \backslash H) \subseteq \sigma_\beta$-$P_S Cl(B) \cap f(X \backslash H) \subseteq \sigma_\beta$-$P_S Cl(B) \cap (Y \backslash B)$. This implies that $f(C) = \phi$, and hence $C = \phi$. So τ_γ-$P_S Cl(f^{-1}(B)) \subseteq H$. Therefore, $f^{-1}(B)$ is γ-P_S-g-closed in (X, τ). □

Theorem 5.17. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be an injective, (γ, β)-P_S-irresolute and (γ, β)-P_S-closed function. If (Y, σ) is β-P_S-$T_{\frac{1}{2}}$, then (X, τ) is γ-P_S-$T_{\frac{1}{2}}$.*

Proof. Let G be any γ-P_S-g-closed set of (X, τ). Since f is (γ, β)-P_S-irresolute and (γ, β)-P_S-closed function. Then by Theorem 5.16 (1), $f(G)$ is β-P_S-g-closed in (Y, σ). Since (Y, σ) is β-P_S-$T_{\frac{1}{2}}$, then $f(G)$ is β-P_S-closed in Y. Again, since f is (γ, β)-P_S-irresolute, then by Theorem 4.1, $f^{-1}(f(G))$ is γ-P_S-closed in X. Hence G is γ-P_S-closed in X since f is injective. Therefore, a space(X, τ) is γ-P_S-$T_{\frac{1}{2}}$. □

Theorem 5.18. *Let a functionf$\colon (X, \tau) \to (Y, \sigma)$ be a surjective, (γ, β)-P_S-irresolute and (γ, β)-P_S-closed. If (X, τ) is γ-P_S-$T_{\frac{1}{2}}$, then (Y, σ) is β-P_S-$T_{\frac{1}{2}}$.*

Proof. Let H be a β-P_S-g-closed set of (Y, σ). Since a function f is (γ, β)-P_S-irresolute and (γ, β)-P_S-closed. Then by Theorem 5.16 (2), $f^{-1}(H)$ is γ-P_S-g-closed in (X, τ). Since (X, τ) is γ-P_S-$T_{\frac{1}{2}}$, then we have, $f^{-1}(H)$ is γ-P_S-closed set in X. Again, since f is (γ, β)-P_S-closed function, then $f(f^{-1}(H))$ is β-P_S-closed in Y. Therefore, H is β-P_S-closed in Y since f is surjective. Hence (Y, σ) is β-P_S-$T_{\frac{1}{2}}$ space. □

Remark 5.19. Every β-P_S-open (resp., β-P_S-closed) function is $(\gamma, \beta P_S)$-open (resp., $(\gamma, \beta P_S)$-closed), but the converse is not true as it is shown in the following example.

Example 5.20. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{c\}, \{b, c\}, X\}$ and $Y = \{1, 2, 3\}$ with the topology $\sigma = \{\phi, Y, \{2\}, \{1, 3\}\}$. Define operations β on σ by $\beta(B) = B$ for all $B \in \sigma$ and γ on τ as follows: For every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{c\} \\ X & \text{otherwise} \end{cases}$$

Then $\tau_\gamma = \{\phi, X, \{c\}\}$ and hence τ_γ-$P_S O(X) = \{\phi, X\}$.
Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function defined as follows:

$$f(x) = \begin{cases} 3 & \text{if } x = a \\ 1 & \text{if } x = b \\ 2 & \text{if } x = c \end{cases}$$

So f is $(\gamma, \beta P_S)$-open (resp., $(\gamma, \beta P_S)$-closed) function, but f is not β-P_S-open (resp., β-P_S-closed) since $\{c\} \in \tau$, but $f(\{b, c\}) = \{1, 2\}$ is not β-P_S-open set in (Y, σ). Again since $\{a\}$ is closed set in (X, τ), but $f(\{a\}) = \{3\}$ is not β-P_S-closed set in (Y, σ).

Theorem 5.21. *Let (X, τ) be γ-semiT_1 space and (Y, σ) be β-semiT_1 space. A function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-closed if and only if f is (γ, β)-preclosed.*

Proof. This is an immediate consequence of Theorem 2.9 (2). ☐

Theorem 5.22. *Let (X, τ) be γ-locally indiscrete space and $f\colon (X, \tau) \to (Y, \sigma)$ be a function, then the following properties of f are equivalent:*

1. *(γ, β)-P_S-open.*
2. *(γ, β)-P_S-closed.*
3. *$(\gamma, \beta P_S)$-closed.*
4. *$(\gamma, \beta P_S)$-open.*

Proof. Follows directly from Theorem 2.9 (1). ☐

Theorem 5.23. *Let (X, τ) be γ-regular space. A function $f\colon (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta P_S)$-open (resp., $(\gamma, \beta P_S)$-closed) if and only if f is β-P_S-open (resp., β-P_S-closed).*

Proof. This is an immediate consequnce of Theorem 2.9 (5). ☐

Theorem 5.24. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be a surjective function. If (X, τ) is γ-hyperconnected space, then f is (γ, β)-P_S-open.*

Proof. This is an immediate consequence of Theorem 2.9 (3). ☐

Theorem 5.25. *A function $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-closed if and only if for each subset S of Y and each γ-P_S-open set O in X containing $f^{-1}(S)$, there exists a β-P_S-open set R in Y containing S such that $f^{-1}(R) \subseteq O$.*

Proof. Suppose that f is (γ, β)-P_S-closed function and let O be a γ-P_S-open set in X containing $f^{-1}(S)$, where S is any subset in Y. Then $f(X \backslash O)$ is β-P_S-open set in Y. If we put $R = Y \backslash f(X \backslash O)$. Then R is β-P_S-closed set in Y such that $S \subseteq R$ and $f^{-1}(R) \subseteq O$.

Conversely, let F be a γ-P_S-closed set in X. Let $S = Y \backslash f(F) \subseteq Y$. Then $f^{-1}(S) \subseteq X \backslash F$ and $X \backslash F$ is γ-P_S-open set in X. By hypothesis, there exists a β-P_S-open set R in Y such that $S = Y \backslash f(F) \subseteq R$ and $f^{-1}(R) \subseteq X \backslash F$. For $f^{-1}(R) \subseteq X \backslash F$ implies $R \subseteq f(X \backslash F) \subseteq Y \backslash f(F)$. Hence $R = Y \backslash f(F)$. Since R is β-P_S-open set in Y. Then $f(F)$ is β-P_S-closed set in Y. Therefore, f is (γ, β)-P_S-closed function. □

Theorem 5.26. *A function $f : (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta P_S)$-closed if and only if for each subset S of Y and each γ-open set O in X containing $f^{-1}(S)$, there exists a β-P_S-open set R in Y containing S such that $f^{-1}(R) \subseteq O$.*

Proof. The proof is similar to Theorem 5.25. □

Definition 5.27. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (γ, β)-P_S-homeomorphism, if f is bijective, (γ, β)-P_S-irresolute and f^{-1} is (γ, β)-P_S-irresolute.

Theorem 5.28. *For a bijective function $f : (X, \tau) \to (Y, \sigma)$. The following properties of f are equivalent:*

1. *f^{-1} is (γ, β)-P_S-irresolute.*
2. *f is (γ, β)-P_S-open.*
3. *f is (γ, β)-P_S-closed.*

Proof. Obvious. □

Theorem 5.29. *The following conditions of f are equivalent for a bijective function $f : (X, \tau) \to (Y, \sigma)$:*

1. *f is (γ, β)-P_S-homeomorphism.*
2. *f is (γ, β)-P_S-irresolute and (γ, β)-P_S-open.*
3. *f is (γ, β)-P_S-irresolute and (γ, β)-P_S-closed.*
4. *$f(\tau_\gamma\text{-}P_S Cl(A)) = \sigma_\beta\text{-}P_S Cl(f(A))$ for each subset A of X.*

Proof. Straightforward. □

Proposition 5.30. *Assume that a function $f : (X, \tau) \to (Y, \sigma)$ is a (γ, β)-P_S-homeomorphism. If (X, τ) is γ-P_S-$T_{\frac{1}{2}}$, then (Y, σ) is β-P_S-$T_{\frac{1}{2}}$.*

Proof. Let $\{y\}$ be any singleton set of (Y, σ). Then there exists an element x of X such that $y = f(x)$. So by hypothesis and Theorem 2.9 (4), we have $\{x\}$ is γ-P_S-closed or γ-P_S-open set in X. By using Theorem 4.1, $\{y\}$ is β-P_S-closed or β-P_S-open set. Then by Theorem 2.9 (4), (Y, σ) is β-P_S-$T_{\frac{1}{2}}$ space. □

In the end of this section, we shall obtain some conditions under which the composition of two functions is (γ, β)-P_S-irresolute and (γ, β)-P_S-continuous.

Theorem 5.31. *Let α be an operation on the topological space (Z, ρ). Let $f\colon (X, \tau) \to (Y, \sigma)$ and $g\colon (Y, \sigma) \to (Z, \rho)$ be functions. Then the composition function $g \circ f\colon (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-continuous if f and g satisfy one of the following conditions:*

1. *f is (γ, β)-P_S-irresolute and g is (β, α)-P_S-continuous.*
2. *f is (γ, β)-P_S-irresolute and g is β-P_S-continuous.*

Proof. (1) Let V be an α-open subset of (Z, ρ). Since g is (β, α)-P_S-continuous, then $g^{-1}(V)$ is β-P_S-open in (Y, σ). Since f is (γ, β)-P_S-irresolute, then by Theorem 4.1, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is γ-P_S-open in X. Therefore, $g \circ f$ is (γ, α)-P_S-continuous.

(2) The proof follows directly from the part (1) since every β-P_S-continuous is (β, α)-P_S-continuous. □

Theorem 5.32. *Let α be an operation on the topological space (Z, ρ). If the functions $f\colon (X, \tau) \to (Y, \sigma)$ is (γ, β)-P_S-irresolute and $g\colon (Y, \sigma) \to (Z, \rho)$ is (β, α)-P_S-irresolute. Then $g \circ f\colon (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-irresolute.*

Proof. It is clear. □

Theorem 5.33. *Let $f\colon (X, \tau) \to (Y, \sigma)$ and $g\colon (Y, \sigma) \to (Z, \rho)$ be functions. Then the composition function $g \circ f\colon (X, \tau) \to (Z, \rho)$ is γ-P_S-continuous if f and g satisfy one of the following conditions:*

1. *f is (γ, β)-P_S-irresolute and g is β-P_S-continuous.*
2. *f is (γ, β)-P_S-continuous and g is β-continuous.*

Proof. The proof is similar to Theorem 5.31. □

Proposition 5.34. *Let $f\colon (X, \tau) \to (Y, \sigma)$ and $g\colon (Y, \sigma) \to (Z, \rho)$ be any functions and α be an operation on ρ. Then the following are holds:*

1. *If f is β-open (β-closed) and g is (β, α)-P_S-open ((β, α)-P_S-closed), then $g \circ f\colon (X, \tau) \to (Z, \rho)$ is α-P_S-open (α-P_S-closed).*

2. *If f is (γ, β)-P_S-open ((γ, β)-P_S-closed) and g is (β, α)-P_S-open ((β, α)-P_S-closed), then $g \circ f : (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-open ((γ, α)-P_S-closed).*

3. *If f is $(\gamma, \beta P_S)$-open ($(\gamma, \beta P_S)$-closed) and g is (β, α)-P_S-open ((β, α)-P_S-closed), then $g \circ f : (X, \tau) \to (Z, \rho)$ is $(\gamma, \alpha P_S)$-open ($(\gamma, \alpha P_S)$-closed).*

Proof. It is clear. □

Proposition 5.35. *If $f : (X, \tau) \to (Y, \sigma)$ is a function, $g : (Y, \sigma) \to (Z, \rho)$ is (β, α)-P_S-open and injective, and $g \circ f : (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-irresolute. Then f is (γ, β)-P_S-irresolute.*

Proof. Let V be an β-P_S-open subset of Y. Since g is (β, α)-P_S-open, $g(V)$ is α-P_S-open subset of Z. Since $g \circ f$ is (γ, α)-P_S-irresolute and g is injective, then $f^{-1}(V) = f^{-1}(g^{-1}(g(V)) = (g \circ f)(g(V))$ is γ-P_S-open in X, which proves that f is (γ, β)-P_S-irresolute. □

Proposition 5.36. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ be any functions and α be an operation on ρ. Then the following are holds:*

1. *If f is (γ, β)-P_S-open and surjective, and $g \circ f : (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-irresolute, then g is (γ, β)-P_S-irresolute.*

2. *If f is (γ, β)-P_S-irresolute and surjectiv, and $g \circ f : (X, \tau) \to (Z, \rho)$ is (γ, α)-P_S-open, then g is (β, α)-P_S-open.*

Proof. Similar to Proposition 5.35. □

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