γ-Ps-Generalized Closed Sets

and γ-Ps-T12 Spaces

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Abstract: This paper defines new class of sets called γ-Ps-generalized closed using γ-Ps-open set and τγ-Ps-closure of a set in a topological space. By using this new set, we introduce a new space called γ-Ps-T12 and define three functions namely γ-Ps-g-continuous, γ-Ps-g-closed and γ-Ps-g-open. Some theoretical results and properties for this space and these functions are obtained. Several examples are given to illustrate some of the results.

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Key Words: γ-Ps-open set, γ-Ps-closed set, γ-Ps-g-closed set, γ-Ps-T12 space, γ-Ps-g-continuous function

1. Introduction

Kasahara [8] defined the concept of α-closed graphs of an operation on τ. Later, Ogata [11] renamed the operation α as γ operation on τ. He defined γ-open sets and introduced the notion of τγ which is the class of all γ-open sets in a topological space (X, τ). Further study by Krishnan and Balachandran ([9], [10]) defined two types of sets called γ-preopen and γ-semiopen sets. The notion of α-γ-open sets have been defined by Kalaivani and Krishnan [7]. Meanwhile, Basu, Afsan and Ghosh [4] defined γ-β-open sets by using the operation γ on...
τ. Carpintero, Rajesh and Rosas [5] introduced another notion of γ-open set called γ-b-open sets of a topological space (X, τ). Recently, Asaad, Ahmad and Omar [1] defined the notion of γ-regular-open sets which lies strictly between the classes of γ-open set and γ-clopen set. They also introduced a new class of sets called γ-P_S-open sets, and they also defined γ-P_S-operations and their properties [2].

In the present paper, we define a new class of sets called γ-P_S-generalized closed using γ-P_S-open set and τγ-P_S-closure of a set and then investigate some of its properties. A new space called γ-P_S-T1 and functions called γ-P_S-continuous, γ-P_S-g-closed and γ-P_S-g-open are defined. Some theorems and results for this space and these functions are obtained.

2. Preliminaries and Basic Definitions

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms assumed unless explicitly stated. An operation γ on the topology τ on X is a mapping γ: τ → P(X) such that U ⊆ γ(U) for each U ∈ τ, where P(X) is the power set of X and γ(U) denotes the value of γ at U [11]. A nonempty subset A of a topological space (X, τ) with an operation γ on τ is said to be γ-open [11] if for each x ∈ A, there exists an open set U containing x such that γ(U) ⊆ A. The complement of a γ-open set is called a γ-closed. The τγ-closure of a subset A of X with an operation γ on τ is defined as the intersection of all γ-closed sets containing A and it is denoted by τγ-Cl(A) [11], and the τγ-interior of a subset A of X with an operation γ on τ is defined as the union of all γ-open sets containing A [10]. Now we begin to recall some known notions which are useful in the sequel.

Definition 2.1. Let (X, τ) be a topological space and γ be an operation on τ. A subset A of X is said to be:

1. γ-regular-open if A = τγ-Int(τγ-Cl(A)) [1].
2. γ-preopen if A ⊆ τγ-Int(τγ-Cl(A)) [9].
3. γ-semiopen if A ⊆ τγ-CI(τγ-Int(A)) [10].
4. α-γ-open if A ⊆ τγ-Int(τγ-Cl(τγ-Int(A))) [7].
5. γ-b-open if A ⊆ τγ-CI(τγ-Int(A)) ∪ τγ-Int(τγ-Cl(A)) [5].
6. γ-β-open if A ⊆ τγ-CI(τγ-Int(τγ-Cl(A))) [4].
7. $\gamma$-clopen if it is both $\gamma$-open and $\gamma$-closed.

8. $\gamma$-dense if $\tau_\gamma Cl(A) = X$ [6].

Definition 2.2. The complement of $\gamma$-regular-open, $\gamma$-preopen, $\gamma$-semiopen, $\alpha$-$\gamma$-open, $\gamma$-$b$-open and $\gamma$-$\beta$-open set is said to be $\gamma$-regular-closed [4], $\gamma$-preclosed [9], $\gamma$-semiclosed [10], $\alpha$-$\gamma$-closed [7], $\gamma$-$b$-closed [5] and $\gamma$-$\beta$-closed [4], respectively.

Definition 2.3. [2] A $\gamma$-preopen subset $A$ of a topological space $(X, \tau)$ is called $\gamma$-$PS$-open if for each $x \in A$, there exists a $\gamma$-semiclosed set $F$ such that $x \in F \subseteq A$. The complement of a $\gamma$-$PS$-open set is called a $\gamma$-$PS$-closed.

The class of all $\gamma$-$PS$-open and $\gamma$-$PS$-closed subsets of a topological space $(X, \tau)$ are denoted by $\tau_\gamma PS O(X)$ and $\tau_\gamma PS C(X)$, respectively.

Definition 2.4. Let $A$ be any subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $\tau$. Then:

1. the $\tau_\gamma PS$-interior of $A$ is defined as the union of all $\gamma$-$PS$-open sets of $X$ contained in $A$ and it is denoted by $\tau_\gamma PS Int(A)$ [2].

2. the $\tau_\gamma PS$-closure, $\tau_\gamma$-preclosure and $\tau_\alpha$-$\gamma$-closure of $A$ is defined as the intersection of all $\gamma$-$PS$-closed, $\gamma$-preclosed and $\alpha$-$\gamma$-closed sets of $X$ containing $A$ and it is denoted by $\tau_\gamma PS Cl(A)$ [2], $\tau_\gamma pCl(A)$ [9] and $\tau_\alpha$-$\gamma$-$Cl(A)$ [7], respectively.

Remark 2.5. [2] Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. For any subset $A$ of a space $X$. The following statements are true.

1. $A$ is $\gamma$-$PS$-closed if and only if $\tau_\gamma PS Cl(A) = A$.

2. $A$ is $\gamma$-$PS$-open if and only if $\tau_\gamma PS Int(A) = A$.

3. $\tau_\gamma PS Cl(X \setminus A) = X \setminus \tau_\gamma PS Int(A)$ and $\tau_\gamma PS Int(X \setminus A) = X \setminus \tau_\gamma PS Cl(A)$.

Remark 2.6. [2] Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. For each element $x \in X$, the set $\{x\}$ is $\gamma$-$PS$-open if and only if $\{x\}$ is $\gamma$-regular-open.

Theorem 2.7. Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $\tau$. Then:

1. $x \in \tau_\gamma PS Cl(A)$ if and only if $A \cap U \neq \emptyset$ for every $\gamma$-$PS$-open set $U$ of $X$ containing $x$ [2].
2. \( x \in \tau_{\gamma}\text{-}p\text{Cl}(A) \) if and only if \( A \cap U \neq \emptyset \) for every \( \gamma \)-preopen set \( U \) of \( X \) containing \( x \) [9].

**Definition 2.8.** Let \((X, \tau)\) be a topological space and \( \gamma \) be an operation on \( \tau \). A subset \( A \) of \( X \) is called:

1. \( \gamma \)-pre-generalized closed (\( \gamma \)-preg-closed) if \( \tau_{\gamma}\text{-}p\text{Cl}(A) \subseteq G \) whenever \( A \subseteq G \) and \( G \) is a \( \gamma \)-preopen set in \( X \) [9].

2. \( \alpha \)-\( \gamma \)-generalized closed (\( \alpha \)-\( \gamma \)-g-closed) if \( \tau_{\alpha\gamma}\text{-}Cl(A) \subseteq G \) whenever \( A \subseteq G \) and \( G \) is a \( \alpha \)-\( \gamma \)-open set in \( X \) [7].

**Definition 2.9.** [9] A topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \) is said to be \( \gamma \)-pre\( T_{1/2} \) if every \( \gamma \)-pre\( g \)-closed set in \( X \) is \( \gamma \)-preclosed.

**Theorem 2.10.** [9] For any topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \). Then \( X \) is \( \gamma \)-pre\( T_{1/2} \) if and only if for each element \( x \in X \), the set \( \{x\} \) is \( \gamma \)-preclosed or \( \gamma \)-preopen.

**Definition 2.11.** [3] Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and \( \gamma \) be an operation on \( \tau \). A function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is called \( \gamma \)-\( P_S \)-continuous if the inverse image of every closed set in \( Y \) is \( \gamma \)-\( P_S \)-closed set in \( X \).

**Definition 2.12.** [3] Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and \( \gamma \) be an operation on \( \tau \). A function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is called \( \gamma \)-\( P_S \)-closed if the image of every closed set in \( Y \) is \( \gamma \)-\( P_S \)-closed set in \( X \).

### 3. \( \gamma \)-\( P_S \)-Generalized Closed Sets

In this section, we define a new class of sets called \( \gamma \)-\( P_S \)-generalized closed using \( \gamma \)-\( P_S \)-open set and \( \tau_{\gamma}\text{-}P_S \)-closure of set. Also study some of its basic properties.

**Definition 3.1.** Let \( A \) be any subset of a topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \) be \( \gamma \)-\( P_S \)-generalized closed (\( \gamma \)-\( P_S \)-g-closed) if \( \tau_{\gamma}\text{-}P_S\text{Cl}(A) \subseteq G \) whenever \( A \subseteq G \) and \( G \) is a \( \gamma \)-\( P_S \)-open set in \( X \).

The class of all \( \gamma \)-\( P_S \)-g-closed sets of \( X \) is denoted by \( \tau_{\gamma}\text{-}P_SGC(X) \).

A set \( A \) is said to be \( \gamma \)-\( P_S \)-generalized open (\( \gamma \)-\( P_S \)-g-open) if its complement is \( \gamma \)-\( P_S \)-g-closed. Or equivalently, a set \( A \) is \( \gamma \)-\( P_S \)-g-open if \( F \subseteq \tau_{\gamma}\text{-}P_S\text{Int}(A) \) whenever \( F \subseteq A \) and \( F \) is a \( \gamma \)-\( P_S \)-closed set in \( X \).

**Lemma 3.2.** Every \( \gamma \)-\( P_S \)-closed set is \( \gamma \)-\( P_S \)-g-closed.
Proof. Let $A$ be any $\gamma$-$P_S$-closed set in a space $X$ and $A \subseteq G$ where $G$ is a $\gamma$-$P_S$-open set in $X$. Then $\gamma$-$P_S Cl(A) \subseteq G$ since $A$ is $\gamma$-$P_S$-closed set. Therefore, $A$ is $\gamma$-$P_S$-$g$-closed set.

The following example shows that the converse of the Lemma 3.2 is not true.

**Example 3.3.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Define an operation $\gamma : \tau \to P(X)$ by $\gamma(A) = A$ for all $A \in \tau$. Then $\gamma$-$P_S Cl(X) = \{\phi, \{a\}, \{b, c\}, X\}$ and $\gamma$-$P_S GC(X) = \{\phi, \{a\}, \{b, c\}, X\}$ and $\gamma$-$P_S GC(X) = \{\phi, \{a\}, \{b, c\}, X\}$ and $\gamma$-$P_S GC(X) = \{\phi, \{a\}, \{b, c\}, X\}$. So $\{b\}$ is $\gamma$-$P_S$-$g$-closed, but it is not $\gamma$-$P_S$-closed.

**Lemma 3.4.** The union of any two $\gamma$-$P_S$-$g$-closed sets may not be $\gamma$-$P_S$-$g$-closed.

**Lemma 3.5.** The intersection of any two $\gamma$-$P_S$-$g$-closed sets may not be $\gamma$-$P_S$-$g$-closed.

Reverse implication of the above theorem does not hold as seen from the following example.

**Example 3.6.** Let $X = (0, 1)$ and $\tau$ be the usual topology on $X$. Define an operation $\gamma$ on $\tau$ by $\gamma(U) = U$ for all $U \in \tau$. Let $A$ be the set of rational numbers in $X$ except the singleton set $\{\frac{1}{2}\}$ and $B$ be the set of irrational numbers in $X$. Then $A$ and $B$ are both $\gamma$-$P_S$-$g$-closed sets, but $A \cup B$ is not $\gamma$-$P_S$-$g$-closed.

**Theorem 3.7.** Let $A$ be a subset of topological space $(X, \tau)$ and $\gamma$ be an operation on $\tau$. Then $A$ is $\gamma$-$P_S$-$g$-closed if and only if $\gamma$-$P_S Cl(A) \setminus A$ does not contain any non-empty $\gamma$-$P_S$-closed set.

Proof. Let $F$ be a non-empty $\gamma$-$P_S$-closed set in $X$ such that $F \subseteq \gamma$-$P_S Cl(A) \setminus A$. Then $F \subseteq X \setminus A$ implies $A \subseteq X \setminus F$. Since $X \setminus F$ is $\gamma$-$P_S$-open set and $A$ is $\gamma$-$P_S$-$g$-closed set, then $\gamma$-$P_S Cl(A) \subseteq X \setminus F$. That is $F \subseteq X \setminus \gamma$-$P_S Cl(A)$. Hence $F \subseteq X \setminus \gamma$-$P_S Cl(A) \cap \gamma$-$P_S Cl(A) \setminus A \subseteq X \setminus \gamma$-$P_S Cl(A) \cap \gamma$-$P_S Cl(A) = \phi$. This shows that $F = \phi$. This is contradiction. Therefore, $F \not\subseteq \gamma$-$P_S Cl(A) \setminus A$.

Conversely, let $A \subseteq G$ and $G$ is $\gamma$-$P_S$-open set in $X$. So $X \setminus G$ is $\gamma$-$P_S$-closed set in $X$. Suppose that $\gamma$-$P_S Cl(A) \not\subseteq G$, then $\gamma$-$P_S Cl(A) \cap X \setminus G$ is a non-empty $\gamma$-$P_S$-closed set such that $\gamma$-$P_S Cl(A) \cap X \setminus G \subseteq \gamma$-$P_S Cl(A) \setminus A$. Contradiction of hypothesis. Hence $\gamma$-$P_S Cl(A) \subseteq G$ and so $A$ is $\gamma$-$P_S$-$g$-closed set.

**Corollary 3.8.** Let $A$ be a $\gamma$-$P_S$-$g$-closed subset of topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$. Then $A$ is $\gamma$-$P_S$-closed if and only if $\gamma$-$P_S Cl(A) \setminus A$...
is \(\gamma\)-\(P_S\)-closed set.

**Proof.** Let \(A\) be a \(\gamma\)-\(P_S\)-closed set. Then \(\tau_{\gamma\cdot P_S}Cl(A) = A\) and hence \(\tau_{\gamma\cdot P_S}Cl(A) \setminus A = \emptyset\) which is \(\gamma\)-\(P_S\)-closed set.

Conversely, suppose \(\tau_{\gamma\cdot P_S}Cl(A) \setminus A\) is \(\gamma\)-\(P_S\)-closed and \(A\) is \(\gamma\)-\(P_S\)-\(g\)-closed. Then by Theorem 3.7, \(\tau_{\gamma\cdot P_S}Cl(A) \setminus A\) does not contain any non-empty \(\gamma\)-\(P_S\)-closed set and since \(\tau_{\gamma\cdot P_S}Cl(A) \setminus A\) is \(\gamma\)-\(P_S\)-closed subset of itself, then \(\tau_{\gamma\cdot P_S}Cl(A) \setminus A = \emptyset\) implies \(\tau_{\gamma\cdot P_S}Cl(A) \cap X \setminus A = \emptyset\). This implies that \(\tau_{\gamma\cdot P_S}Cl(A) = A\). Therefore, \(A\) is \(\gamma\)-\(P_S\)-closed in \(X\).

**Remark 3.9.** For any set \(A \subseteq (X, \tau)\), \(\tau_{\gamma\cdot P_S}Int(\tau_{\gamma\cdot P_S}Cl(A) \setminus A) = \emptyset\).

**Proof.** Obvious. \(\square\)

**Theorem 3.10.** In any topological space \((X, \tau)\), a set \(A \subseteq (X, \tau)\) is \(\gamma\)-\(P_S\)-\(g\)-closed if and only if \(\tau_{\gamma\cdot P_S}Cl(A) \setminus A\) is \(\gamma\)-\(P_S\)-\(g\)-open set.

**Proof.** Let \(F\) be a \(\gamma\)-\(P_S\)-closed set in \(X\) such that \(F \subseteq \tau_{\gamma\cdot P_S}Cl(A) \setminus A\). Since \(A\) is \(\gamma\)-\(P_S\)-\(g\)-closed, then by Theorem 3.7, \(F = \emptyset\). Hence \(F \subseteq \tau_{\gamma\cdot P_S}Int(\tau_{\gamma\cdot P_S}Cl(A) \setminus A)\). This shows that \(\tau_{\gamma\cdot P_S}Cl(A) \setminus A\) is \(\gamma\)-\(P_S\)-\(g\)-open set.

Conversely, suppose that \(A \subseteq G\), where \(G\) is a \(\gamma\)-\(P_S\)-open set in \(X\). So \(\tau_{\gamma\cdot P_S}Cl(A) \cap X \setminus G \subseteq \tau_{\gamma\cdot P_S}Cl(A) \cap X \setminus A = \tau_{\gamma\cdot P_S}Cl(A) \setminus A\). Since \(X \setminus G\) is a \(\gamma\)-\(P_S\)-closed and hence \(\tau_{\gamma\cdot P_S}Cl(A) \cap X \setminus G\) is \(\gamma\)-\(P_S\)-closed set in \(X\) and \(\tau_{\gamma\cdot P_S}Cl(A) \setminus A\) is \(\gamma\)-\(P_S\)-\(g\)-open set. Then \(\tau_{\gamma\cdot P_S}Cl(A) \cap X \setminus G \subseteq \tau_{\gamma\cdot P_S}Int(\tau_{\gamma\cdot P_S}Cl(A) \setminus A) = \phi\). By Remark 3.9, \(\tau_{\gamma\cdot P_S}Int(\tau_{\gamma\cdot P_S}Cl(A) \setminus A) = \phi\) implies that \(\tau_{\gamma\cdot P_S}Cl(A) \cap X \setminus G = \phi\) and hence \(\tau_{\gamma\cdot P_S}Cl(A) \subseteq G\). This means that \(A\) is \(\gamma\)-\(P_S\)-\(g\)-closed. \(\square\)

**Theorem 3.11.** Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). If a subset \(A\) of \(X\) is \(\gamma\)-\(P_S\)-\(g\)-closed and \(\gamma\)-\(P_S\)-open, then \(A\) is \(\gamma\)-\(P_S\)-closed.

**Proof.** Since \(A\) is \(\gamma\)-\(P_S\)-\(g\)-closed and \(\gamma\)-\(P_S\)-open set in \(X\), then \(\tau_{\gamma\cdot P_S}Cl(A) \subseteq A\) and so \(A\) is \(\gamma\)-\(P_S\)-closed. \(\square\)

**Theorem 3.12.** Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). If a subset \(A\) of \(X\) is \(\gamma\)-\(P_S\)-\(g\)-closed and \(\gamma\)-\(P_S\)-open and \(F\) is \(\gamma\)-\(P_S\)-closed, then \(A \cap F\) is \(\gamma\)-\(P_S\)-closed.

**Proof.** Since \(A\) is both \(\gamma\)-\(P_S\)-\(g\)-closed and \(\gamma\)-\(P_S\)-open set. Then by Theorem 3.11, \(A\) is \(\gamma\)-\(P_S\)-closed and since \(F\) is \(\gamma\)-\(P_S\)-closed, then \(A \cap F\) is \(\gamma\)-\(P_S\)-closed. \(\square\)

**Corollary 3.13.** If \(A \subseteq X\) is both \(\gamma\)-\(P_S\)-\(g\)-closed and \(\gamma\)-\(P_S\)-open and \(F\) is \(\gamma\)-\(P_S\)-closed, then \(A \cap F\) is \(\gamma\)-\(P_S\)-\(g\)-closed.
Proof. Follows from Theorem 3.12 and the fact that every \( \gamma \)-\( P_S \)-closed set is \( \gamma \)-\( P_S \)-\( g \)-closed.

\[ \square \]

**Corollary 3.14.** For any topological space \((X, \tau)\). If a subset \( A \) of \( X \) is \( \gamma \)-\( P_S \)-\( g \)-closed and \( \gamma \)-\( P_S \)-open, then \( A \) is \( \gamma \)-\( \text{preg} \)-closed.

Proof. The proof follows directly from Theorem 3.11 and the fact that every \( \gamma \)-\( P_S \)-closed set is \( \gamma \)-\( \text{preclo} \)sed and every \( \gamma \)-\( \text{preclo} \)sed set is \( \gamma \)-\( \text{preg} \)-closed [9].

\[ \square \]

**Theorem 3.15.** If \( A \subseteq (X, \tau) \) is both \( \gamma \)-regular-open and \( \gamma \)-\( P_S \)-\( g \)-closed, then \( A \) is \( \gamma \)-regular-closed and hence it is \( \gamma \)-clopen.

Proof. Let \( A \) be both \( \gamma \)-regular-open and \( \gamma \)-\( P_S \)-\( g \)-closed. Since \( A \) is \( \gamma \)-regular-open set. Then \( A \) is \( \gamma \)-\( P_S \)-open and by Theorem 3.11, \( A \) is \( \gamma \)-\( P_S \)-closed and so it is \( \gamma \)-preclosed. Again since \( A \) is \( \gamma \)-regular-open set, then \( A \) is \( \gamma \)-semiopen. Therefore, \( A \) is \( \gamma \)-regular-closed in \( X \). Thus \( A \) is both \( \gamma \)-open and \( \gamma \)-closed and hence it is \( \gamma \)-clopen.

\[ \square \]

**Lemma 3.16.** For any subset \( A \) in \((X, \tau)\). If \( A \) is \( \gamma \)-semiopen, then \( \tau_\gamma \)-\( P_S \)\( Cl(A) = \tau_\gamma \)-\( pCl(A) \).

Proof. Let \( x \notin \tau_\gamma \)-\( P_S \)\( Cl(A) \), then there exists a \( \gamma \)-preopen set \( U \) containing \( x \) such that \( A \cap U = \phi \) implies that \( \tau_\gamma \)-\( Cl(\tau_\gamma \text{-Int}(A)) = \tau_\gamma \text{-Int}(\tau_\gamma \text{-Cl}(U)) = \phi \). Since \( A \) is \( \gamma \)-semiopen, then \( A \cap \tau_\gamma \text{-Int}(\tau_\gamma \text{-Cl}(U)) = \phi \). Since \( U \) is \( \gamma \)-preopen set containing \( x \), then \( x \in \tau_\gamma \text{-Int}(\tau_\gamma \text{-Cl}(U)) \) and \( \tau_\gamma \text{-Int}(\tau_\gamma \text{-Cl}(U)) \) is \( \gamma \)-\( P_S \)-open set. So by Theorem 2.7 (1), \( x \notin \tau_\gamma \)-\( P_S \)\( Cl(A) \). Hence \( \tau_\gamma \)-\( P_S \)\( Cl(A) \subseteq \tau_\gamma \)-\( pCl(A) \). But \( \tau_\gamma \)-\( pCl(A) \subseteq \tau_\gamma \)-\( P_S \)\( Cl(A) \) in general. Then \( \tau_\gamma \)-\( P_S \)\( Cl(A) = \tau_\gamma \)-\( pCl(A) \).

Similar to Lemma 3.16, we can show that \( \tau_\gamma \)-\( pCl(A) = \tau_\gamma \)-\( Cl(A) = \tau_\alpha \)-\( \gamma \)-\( Cl(A) \) for every \( \gamma \)-semiopen set \( A \) in \((X, \tau)\). So we have the next corollary.

**Corollary 3.17.** For each \( \gamma \)-semiopen \( A \) in \((X, \tau)\), we have

\[
\tau_\gamma \)-\( P_S \)\( Cl(A) = \tau_\gamma \)-\( pCl(A) = \tau_\gamma \)-\( Cl(A) = \tau_\alpha \)-\( \gamma \)-\( Cl(A) \).
\]

**Lemma 3.18.** For any subset \( A \) in \((X, \tau)\). If \( A \) is \( \gamma \)-\( \beta \)-open, then \( \tau_\gamma \)-\( Cl(A) = \tau_\alpha \)-\( \gamma \)-\( Cl(A) \).

Proof. The proof is similar to Lemma 3.16.

\[ \square \]

**Theorem 3.19.** If a subset \( A \) of \((X, \tau)\) is both \( \alpha \)-\( \gamma \)-open and \( \gamma \)-\( \text{preg} \)-closed, then \( A \) is \( \gamma \)-\( P_S \)-\( g \)-closed.
Proof. Suppose that $A$ is both $\alpha$-$\gamma$-open and $\gamma$-preg-$\alpha$-closed set in $X$. Let $A \subseteq G$ and $G$ be a $\gamma$-$P_S$-open set in $X$. Since $A$ is $\alpha$-$\gamma$-open. Then $A$ is $\gamma$-preopen. Now $A \subseteq A$. By hypothesis, $\tau_\gamma pCl(A) \subseteq A$. Again since $A$ is $\alpha$-$\gamma$-open, then $A$ is $\gamma$-semiopen. By Corollary 3.17, we get $\tau_\gamma P_S Cl(A) \subseteq A \subseteq G$. Thus, $A$ is $\gamma$-$P_S$-$g$-closed.

**Theorem 3.20.** If a set $A$ in $X$ is both $\alpha$-$\gamma$-open and $\alpha$-$\gamma$-$g$-closed, then $A$ is $\gamma$-$P_S$-$g$-closed.

Proof. The proof is similar to Theorem 3.19 and using Corollary 3.17 to obtain $\tau_{\alpha-\gamma} Cl(A) = \tau_\gamma P_S Cl(A)$ for every $\gamma$-semiopen set in $X$.

The converse of Theorem 3.19 and Theorem 3.20 are true when $A$ is $\gamma$-regular-open as it can be seen from the following corollary.

**Corollary 3.21.** Let $A$ be a $\gamma$-regular-open subset of a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$. Then the following conditions are equivalent:

1. $A$ is $\gamma$-$P_S$-$g$-closed.
2. $A$ is $\gamma$-preg-$\alpha$-closed.
3. $A$ is $\alpha$-$\gamma$-$g$-closed.

**Theorem 3.22.** In a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$, then every subset of $X$ is $\gamma$-$P_S$-$g$-closed if and only if $\tau_\gamma P_S O(X) = \tau_\gamma P_S C(X)$.

Proof. Assume that every subset of $X$ is $\gamma$-$P_S$-$g$-closed. Let $U \in \tau_\gamma P_S O(X)$. Since $U$ is $\gamma$-$P_S$-$g$-closed. Then by Theorem 3.11, we have $U$ is $\gamma$-$P_S$-closed. Hence $\tau_\gamma P_S O(X) \subseteq \tau_\gamma P_S C(X)$. If $F \in \tau_\gamma P_S C(X)$, then $X \setminus F \in \tau_\gamma P_S O(X)$ and $X \setminus F$ is $\gamma$-$P_S$-$g$-closed. Then by Theorem 3.11, $X \setminus F$ is $\gamma$-$P_S$-closed and hence $F$ is $\gamma$-$P_S$-open set. Thus, $\tau_\gamma P_S C(X) \subseteq \tau_\gamma P_S O(X)$. This means that $\tau_\gamma P_S O(X) = \tau_\gamma P_S C(X)$.

Conversely, suppose that $\tau_\gamma P_S O(X) = \tau_\gamma P_S C(X)$ and that $A \subseteq G$ and $G \in \tau_\gamma P_S O(X)$. Then $\tau_\gamma P_S Cl(A) \subseteq \tau_\gamma P_S Cl(G) = G$. So $A$ is $\gamma$-$P_S$-$g$-closed.

**Theorem 3.23.** Let $A, B$ be subsets of a topological space $(X, \tau)$ and $\gamma$ be an operation on $\tau$. If $A$ is $\gamma$-$P_S$-$g$-closed and $A \subseteq B \subseteq \tau_\gamma P_S Cl(A)$, then $B$ is $\gamma$-$P_S$-$g$-closed set.
Proof. Let \( A \) be any \( \gamma \)-\( P_S \)-\( g \)-closed set in \((X, \tau)\) and \( B \subseteq G \) where \( G \) is \( \gamma \)-\( P_S \)-open. Since \( A \subseteq B \), then \( A \subseteq G \) and hence \( \tau_\gamma P_S \text{Cl}(A) \subseteq G \). Since \( B \subseteq \tau_\gamma P_S \text{Cl}(A) \) implies \( \tau_\gamma P_S \text{Cl}(B) \subseteq \tau_\gamma P_S \text{Cl}(A) \). Thus \( \tau_\gamma P_S \text{Cl}(B) \subseteq G \) and this shows that \( B \) is \( \gamma \)-\( P_S \)-\( g \)-closed set.

From Theorems 3.7 and 3.23, we obtain the following proposition.

Proposition 3.24. Let \( A, B \) be subsets of a topological space \((X, \tau)\) and \( \gamma \) be an operation on \( \tau \). If \( A \) is \( \gamma \)-\( P_S \)-\( g \)-closed and \( A \subseteq B \subseteq \tau_\gamma P_S \text{Cl}(A) \), then \( \tau_\gamma P_S \text{Cl}(B) \backslash B \) contains no non-empty \( \gamma \)-\( P_S \)-\( g \)-closed set.

Theorem 3.25. Let \( A \) and \( B \) be subsets of \((X, \tau)\). If \( A \) is \( \gamma \)-\( P_S \)-\( g \)-open and \( \tau_\gamma P_S \text{Int}(A) \subseteq B \subseteq A \), then \( B \) is \( \gamma \)-\( P_S \)-\( g \)-open set.

Proof. Since \( \tau_\gamma P_S \text{Int}(A) \subseteq B \subseteq A \) implies that \( X \backslash A \subseteq X \backslash B \subseteq X \backslash \tau_\gamma P_S \text{Int}(A) \). By Remark 2.5 (3), we get \( X \backslash A \subseteq X \backslash B \subseteq \tau_\gamma P_S \text{Cl}(X \backslash A) \). Since \( A \) is \( \gamma \)-\( P_S \)-\( g \)-open and then \( X \backslash A \) is \( \gamma \)-\( P_S \)-\( g \)-closed. So by Theorem 3.23, \( X \backslash B \) is \( \gamma \)-\( P_S \)-\( g \)-closed and hence \( B \) is \( \gamma \)-\( P_S \)-\( g \)-open.

Theorem 3.26. A subset \( A \) in \((X, \tau)\) is \( \gamma \)-\( P_S \)-\( g \)-open if and only if \( G = X \) whenever \( G \) is \( \gamma \)-\( P_S \)-open set in \( X \) and \( \tau_\gamma P_S \text{Int}(A) \cup X \backslash A \subseteq G \).

Proof. Let \( G \) be a \( \gamma \)-\( P_S \)-open set in \( X \) and \( \tau_\gamma P_S \text{Int}(A) \cup X \backslash A \subseteq G \). This implies \( X \backslash G \subseteq \tau_\gamma P_S \text{Cl}(X \backslash A) \cap A = \tau_\gamma P_S \text{Cl}(X \backslash A) \backslash (X \backslash A) \). Since \( G \) is \( \gamma \)-\( P_S \)-\( g \)-open and \( A \) is \( \gamma \)-\( P_S \)-\( g \)-open, then \( X \backslash G \) is \( \gamma \)-\( P_S \)-\( g \)-closed and \( X \backslash A \) is \( \gamma \)-\( P_S \)-\( g \)-closed. So by Theorem 3.7, \( X \backslash G = \emptyset \) implies \( G = X \).

Conversely, suppose \( F \) is a \( \gamma \)-\( P_S \)-\( g \)-closed set in \( X \) and \( F \subseteq A \). Then \( X \backslash A \subseteq X \backslash F \) and hence \( \tau_\gamma P_S \text{Int}(A) \cup X \backslash A \subseteq \tau_\gamma P_S \text{Int}(A) \cup X \backslash F \). Since \( \tau_\gamma P_S \text{Int}(A) \cup X \backslash F \) is \( \gamma \)-\( P_S \)-open set in \( X \), then by hypothesis \( \tau_\gamma P_S \text{Int}(A) \cup X \backslash F = X \). It follows that \( F \subseteq \tau_\gamma P_S \text{Int}(A) \). Therefore, \( A \) is \( \gamma \)-\( P_S \)-\( g \)-open set in \( X \).

Recall that a topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \) is \( \gamma \)-\( \text{semi-T}_1 \) if for each pair of distinct points \( x, y \) in \( X \), there exist two \( \gamma \)-semipreopen sets \( U \) and \( V \) such that \( x \in U \) but \( y \notin U \) and \( y \in V \) but \( x \notin V \) [10].

Theorem 3.27. [2] Let \((X, \tau)\) be a topological space and \( \gamma \) be an operation on \( \tau \). If \( X \) is \( \gamma \)-\( \text{semi-T}_1 \), then the notion of \( \gamma \)-\( P_S \)-open set and \( \gamma \)-\( \text{preopen} \) set coincide, or this means that the notion of \( \gamma \)-\( P_S \)-\( \text{closed} \) set and \( \gamma \)-\( \text{preclosed} \) set coincide.

Remark 3.28. In the above theorem, we can conclude that \( \tau_\gamma P_S \text{Cl}(A) = \tau_\gamma p \text{Cl}(A) \) for any subset \( A \) of a \( \gamma \)-\( \text{semi-T}_1 \) space \( X \).
Theorem 3.29. Let \((X, \tau)\) be \(\gamma\)-semi-\(T_1\) space and \(\gamma\) be an operation on \(\tau\). A set \(A\) is \(\gamma\)-\(P_S\)-g-closed if and only if \(A\) is \(\gamma\)-preg-closed.

Proof. Follows from Theorem 3.27 and Remark 3.28. \(\square\)

Recall that a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma\)-locally indiscrete if every \(\gamma\)-open subset of \(X\) is \(\gamma\)-closed, or every \(\gamma\)-closed subset of \(X\) is \(\gamma\)-open [1].

Theorem 3.30. [2] Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). If \(X\) is \(\gamma\)-locally indiscrete, then \(\tau_\gamma-P_SO(X) = \tau_\gamma\).

Lemma 3.31. If a space \((X, \tau)\) is \(\gamma\)-locally indiscrete, then every \(\gamma\)-\(P_S\)-open subset of \(X\) is \(\gamma\)-\(P_S\)-closed.

Proof. Follows directly from Theorem 3.30. \(\square\)

The following theorem shows that if a space \(X\) is \(\gamma\)-locally indiscrete, then \(\tau_\gamma-P_SGC(X)\) is discrete topology.

Theorem 3.32. If a topological space \((X, \tau)\) is \(\gamma\)-locally indiscrete, then every subset of \(X\) is \(\gamma\)-\(P_S\)-g-closed.

Proof. Suppose that \((X, \tau)\) is \(\gamma\)-locally indiscrete space and \(A \subseteq U\) where \(U \in \tau_\gamma-P_SO(X)\). Then \(\tau_\gamma-P_SCl(A) \subseteq \tau_\gamma-P_SCl(U)\) and by Lemma 3.31, we have \(\tau_\gamma-P_SCl(A) \subseteq U\) and so \(A\) is \(\gamma\)-\(P_S\)-g-closed set. \(\square\)

Recall that a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma\)-hyperconnected if every \(\gamma\)-open subset of \(X\) is \(\gamma\)-dense [1].

Theorem 3.33. [2] If a topological space \((X, \tau)\) is \(\gamma\)-hyperconnected if and only if \(\tau_\gamma-P_SO(X) = \{\phi, X\}\).

Theorem 3.34. In a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\), if \(\tau_\gamma-P_SO(X) = \{\phi, X\}\), then every subset of \(X\) is a \(\gamma\)-\(P_S\)-\(g\)-closed.

Proof. Let \(A\) be any subset of a topological space \((X, \tau)\) and \(\tau_\gamma-P_SO(X) = \{\phi, X\}\). Suppose that \(A = \phi\), then \(A\) is a \(\gamma\)-\(P_S\)-\(g\)-closed set in \(X\). If \(A \neq \phi\), then \(X\) is the only \(\gamma\)-\(P_S\)-open set containing \(A\) and hence \(\tau_\gamma-P_SCl(A) \subseteq X\). So \(A\) is a \(\gamma\)-\(P_S\)-\(g\)-closed set in \(X\). \(\square\)

Corollary 3.35. If a topological space \((X, \tau)\) is \(\gamma\)-hyperconnected, then every subset of \(X\) is \(\gamma\)-\(P_S\)-\(g\)-closed.

Proof. Follows from Theorem 3.33 and Theorem 3.34. \(\square\)
Theorem 3.36. In any topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\). For an element \(x \in X\), the set \(X \setminus \{x\}\) is \(\gamma\)-\(P_S\)-g-closed or \(\gamma\)-\(P_S\)-open.

Proof. Suppose that \(X \setminus \{x\}\) is not \(\gamma\)-\(P_S\)-open. Then \(X\) is the only \(\gamma\)-\(P_S\)-open set containing \(X \setminus \{x\}\). This implies that \(\tau(g) \cap P_S Cl(X \setminus \{x\}) \subseteq X\). Thus \(X \setminus \{x\}\) is a \(\gamma\)-\(P_S\)-g-closed set in \(X\).

\[\square\]

Corollary 3.37. In any topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\). For an element \(x \in X\), either the set \(\{x\}\) is \(\gamma\)-\(P_S\)-closed or the set \(X \setminus \{x\}\) is \(\gamma\)-\(P_S\)-g-closed.

Proof. Suppose \(\{x\}\) is not \(\gamma\)-\(P_S\)-closed, then \(X \setminus \{x\}\) is not \(\gamma\)-\(P_S\)-open. Hence by Theorem 3.36, \(X \setminus \{x\}\) is \(\gamma\)-\(P_S\)-g-closed set in \(X\).

\[\square\]

Lemma 3.38. Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). A set \(A\) in \((X, \tau)\) is \(\gamma\)-\(P_S\)-g-closed if and only if \(A \cap \tau(g) \cap P_S Cl(\{x\}) \neq \phi\) for every \(x \in \tau(g) \cap P_S Cl(A)\).

Proof. Suppose \(A\) is \(\gamma\)-\(P_S\)-g-closed set in \(X\) and suppose (if possible) that there exists an element \(x \in \tau(g) \cap P_S Cl(A)\) such that \(A \cap \tau(g) \cap P_S Cl(\{x\}) = \phi\). This follows that \(A \subseteq X \setminus \tau(g) \cap P_S Cl(\{x\})\). Since \(\gamma\)-\(P_S\)-closed implies \(X \setminus \tau(g) \cap P_S Cl(\{x\})\) is \(\gamma\)-\(P_S\)-open and \(A\) is \(\gamma\)-\(P_S\)-g-closed set in \(X\). Then \(\tau(g) \cap P_S Cl(A) \subseteq X \setminus \tau(g) \cap P_S Cl(\{x\})\). This means that \(x \notin \tau(g) \cap P_S Cl(A)\). This is a contradiction. Hence \(A \cap \tau(g) \cap P_S Cl(\{x\}) \neq \phi\).

Conversely, let \(G\) be any \(\gamma\)-\(P_S\)-open set in \(X\) containing \(A\). To show that \(\tau(g) \cap P_S Cl(A) \subseteq G\). Let \(x \in \tau(g) \cap P_S Cl(A)\). Then by hypothesis, \(A \cap \tau(g) \cap P_S Cl(\{x\}) \neq \phi\). So there exists an element \(y \in A \cap \tau(g) \cap P_S Cl(\{x\})\). Thus \(y \in A \subseteq G\) and \(y \in \tau(g) \cap P_S Cl(\{x\})\). By Theorem 2.7 (1), \(\{x\} \cap G \neq \phi\). Hence \(x \in G\) and so \(\tau(g) \cap P_S Cl(A) \subseteq G\). Therefore, \(A\) is \(\gamma\)-\(P_S\)-g-closed set in \((X, \tau)\).

\[\square\]

Theorem 3.39. For any subset \(A\) of a topological space \((X, \tau)\). Then \(A \cap \tau(g) \cap P_S Cl(\{x\}) \neq \phi\) for every \(x \in \tau(g) \cap P_S Cl(A)\) if and only if \(\tau(g) \cap P_S Cl(A) \setminus A\) does not contain any non-empty \(\gamma\)-\(P_S\)-closed set.

Proof. The proof is directly from Theorem 3.7 and Lemma 3.38.

\[\square\]

Corollary 3.40. Let \(A\) be a subset of topological space \((X, \tau)\) and \(\gamma\) be an operation on \(\tau\). Then \(A\) is \(\gamma\)-\(P_S\)-g-closed if and only if \(A = E \setminus F\), where \(E\) is \(\gamma\)-\(P_S\)-closed set and \(F\) contains no non-empty \(\gamma\)-\(P_S\)-closed set.
Proof. Let \( A \) be any \( \gamma \)-\( P_S \)-g-closed set in \((X, \tau)\). Then by Theorem 3.7, \( \tau_\gamma \)-\( P_S Cl(A) \setminus A = F \) contains no non-empty \( \gamma \)-\( P_S \)-closed set. Let \( E = \tau_\gamma \)-\( P_S Cl(A) \) is \( \gamma \)-\( P_S \)-closed set such that \( A = E \setminus F \).

Conversely, let \( A = E \setminus F \), where \( E \) is \( \gamma \)-\( P_S \)-closed set and \( F \) contains no non-empty \( \gamma \)-\( P_S \)-closed set. Let \( A \subseteq G \) and \( G \) is \( \gamma \)-\( P_S \)-open set in \( X \). Then \( E \cap X \setminus G \) is a \( \gamma \)-\( P_S \)-closed subset of \( F \) and hence it is empty. Therefore, \( \tau_\gamma \)-\( P_S Cl(A) \subseteq E \subseteq G \). Thus \( A \) is \( \gamma \)-\( P_S \)-g-closed set. \( \square \)

4. \( \gamma \)-\( P_S-T_{\frac{1}{2}} \) Spaces

This section introduces a new space called \( \gamma \)-\( P_S-T_{\frac{1}{2}} \) by using \( \gamma \)-\( P_S \)-g-closed set.

**Definition 4.1.** A topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \) is said to be \( \gamma \)-\( P_S-T_{\frac{1}{2}} \) if every \( \gamma \)-\( P_S \)-g-closed set in \( X \) is \( \gamma \)-\( P_S \)-closed set.

**Lemma 4.2.** A topological space \((X, \tau)\) is \( \gamma \)-\( P_S-T_{\frac{1}{2}} \) if and only if \( \tau_\gamma \)-\( P_S GC(X) = \tau_\gamma \)-\( P_S C(X) \).

**Proof.** Follows from Definition 4.1 and Lemma 3.2. \( \square \)

**Theorem 4.3.** For any topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \). Then \( X \) is \( \gamma \)-\( P_S-T_{\frac{1}{2}} \) if and only if for each element \( x \in X \), the set \( \{x\} \) is \( \gamma \)-\( P_S \)-closed or \( \gamma \)-\( P_S \)-open.

**Proof.** Let \( X \) be a \( \gamma \)-\( P_S-T_{\frac{1}{2}} \) space and let \( \{x\} \) is not \( \gamma \)-\( P_S \)-closed set in \( X \). By Corollary 3.37, \( X \setminus \{x\} \) is \( \gamma \)-\( P_S \)-g-closed. Since \( X \) is \( \gamma \)-\( P_S-T_{\frac{1}{2}} \), then \( X \setminus \{x\} \) is \( \gamma \)-\( P_S \)-closed set which means that \( \{x\} \) is \( \gamma \)-\( P_S \)-open set in \( X \).

Conversely, let \( F \) be any \( \gamma \)-\( P_S \)-g-closed set in the space \((X, \tau)\). We have to show that \( F \) is \( \gamma \)-\( P_S \)-closed (that is \( \tau_\gamma \)-\( P_S Cl(F) = F \)). Let \( x \in \tau_\gamma \)-\( P_S Cl(F) \). By hypothesis \( \{x\} \) is \( \gamma \)-\( P_S \)-closed or \( \gamma \)-\( P_S \)-open for each \( x \in X \). So we have two cases:

Case (1): If \( \{x\} \) is \( \gamma \)-\( P_S \)-closed set. Suppose \( x \notin F \), then \( x \in \tau_\gamma \)-\( P_S Cl(F) \setminus F \) contains a non-empty \( \gamma \)-\( P_S \)-closed set \( \{x\} \). A contradiction since \( F \) is \( \gamma \)-\( P_S \)-g-closed set and according to the Theorem 3.7. Hence \( x \in F \). This follows that \( \tau_\gamma \)-\( P_S Cl(F) \subseteq F \) and so \( \tau_\gamma \)-\( P_S Cl(F) = F \). This means that \( F \) is \( \gamma \)-\( P_S \)-closed set in \( X \). Thus a space \( X \) is \( \gamma \)-\( P_S-T_{\frac{1}{2}} \).

Case (2): If \( \{x\} \) is \( \gamma \)-\( P_S \)-open set. Then by Theorem 2.7 (1), \( F \cap \{x\} \neq \phi \) which implies that \( x \in F \). So \( \tau_\gamma \)-\( P_S Cl(F) \subseteq F \). Thus \( F \) is \( \gamma \)-\( P_S \)-closed. Therefore, \( X \) is \( \gamma \)-\( P_S-T_{\frac{1}{2}} \) space. \( \square \)
**Proposition 4.4.** If a space \((X, \tau)\) is \(\gamma\)-\(P_S\)-\(T^1_2\), then the set \(\{x\}\) is \(\gamma\)-\(P_S\)-closed or \(\gamma\)-regular-open for each \(x \in X\).

*Proof.* The proof is directly from Theorem 4.3 and Remark 2.6. \(\square\)

**Corollary 4.5.** If a space \((X, \tau)\) is \(\gamma\)-\(P_S\)-\(T^1_2\), then for each point \(x \in X\) the set \(\{x\}\) is \(\gamma\)-\(b\)-closed.

*Proof.* Let \(X\) be a \(\gamma\)-\(P_S\)-\(T^1_2\) space. Then by Theorem 4.3, \(\{x\}\) of \(X\) is either \(\gamma\)-\(P_S\)-closed set or \(\gamma\)-\(P_S\)-open set. If \(\{x\}\) is \(\gamma\)-\(P_S\)-closed set, then \(\{x\}\) is \(\gamma\)-\(b\)-closed set. If \(\{x\}\) is \(\gamma\)-\(P_S\)-open, then by Remark 2.6, \(\{x\}\) is \(\gamma\)-regular-open set and hence it is \(\gamma\)-\(b\)-closed set. In both cases, we have \(\{x\}\) is \(\gamma\)-\(b\)-closed set. \(\square\)

**Theorem 4.6.** Every \(\gamma\)-\(P_S\)-\(T^1_2\) space is \(\gamma\)-\(pre\)\(T^1_2\).

*Proof.* Let \((X, \tau)\) be a \(\gamma\)-\(P_S\)-\(T^1_2\) space. Then by Theorem 4.3, every singleton set is \(\gamma\)-\(P_S\)-closed or \(\gamma\)-\(P_S\)-open. This implies that every singleton set is \(\gamma\)-\(preclosed\) or \(\gamma\)-\(preopen\). Therefore, by Theorem 2.10, \((X, \tau)\) is \(\gamma\)-\(pre\)\(T^1_2\) space. \(\square\)

The converse of the above theorem does not hold as seen from the following example.

**Example 4.7.** Let \(X = \{a, b, c\}\) with the topology
\[
\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}.
\]
Define an operation \(\gamma: \tau \rightarrow P(X)\) as follows: for every \(A \in \tau\)
\[
\gamma(A) = \begin{cases} 
Cl(A) & \text{if } c \notin A \\
A & \text{if } c \in A
\end{cases}
\]
Then \(\tau_\gamma = \{\phi, X, \{c\}, \{b, c\}, \{a, b\}\}\), \(\tau_\gamma\)-\(P_S\)\(C(X) = \{\phi, X, \{b\}, \{c\}, \{a, b\}\}\) and \(\tau_\gamma\)-\(P_S\)\(GC(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}\). Then \((X, \tau)\) is \(\gamma\)-\(pre\)\(T^1_2\) but it is not \(\gamma\)-\(P_S\)-\(g\)-closed, but it is not \(\gamma\)-\(P_S\)-closed.

**Theorem 4.8.** Let \((X, \tau)\) be a \(\gamma\)-\(semi\)-\(T^1_2\) space. Then \((X, \tau)\) is \(\gamma\)-\(P_S\)-\(T^1_2\) if and only if \((X, \tau)\) is \(\gamma\)-\(pre\)\(T^1_2\).

*Proof.* The proof follows from Theorem 3.27 and Theroem 3.29. \(\square\)
5. $\gamma$-$P_S$-$g$-Continuous Functions

In this section, we introduce a new class of functions called $\gamma$-$P_S$-$g$-continuous by using $\gamma$-$P_S$-$g$-closed set. Some theorems and properties for this function are studied.

**Definition 5.1.** Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\gamma$ be an operation on $\tau$. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called $\gamma$-$P_S$-$g$-continuous if the inverse image of every closed set in $Y$ is $\gamma$-$P_S$-$g$-closed set in $X$.

**Theorem 5.2.** For a function $f: (X, \tau) \rightarrow (Y, \sigma)$ with an operation $\gamma$ on $\tau$, the following statements are equivalent:

1. $f$ is $\gamma$-$P_S$-$g$-continuous.
2. The inverse image of every open set in $Y$ is $\gamma$-$P_S$-$g$-open set in $X$.
3. For each point $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a $\gamma$-$P_S$-$g$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$.

Proof. Straightforward.

**Remark 5.3.** Every $\gamma$-$P_S$-continuous function is $\gamma$-$P_S$-$g$-continuous.

Proof. Obvious since every $\gamma$-$P_S$-closed set is $\gamma$-$P_S$-$g$-closed set.

The converse of the above remark does not true as seen from the following example.

**Example 5.4.** Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$ as in Example 4.7. Suppose that $Y = \{1, 2, 3\}$ and $\sigma = \{\phi, Y, \{1\}, \{1, 3\}\}$ be a topology on $Y$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function defined as follows: $f(a) = 1, f(b) = 2$ and $f(c) = 3$. Then $f$ is $\gamma$-$P_S$-$g$-continuous, but $f$ is not $\gamma$-$P_S$-continuous since $\{2, 3\}$ is closed in $(Y, \sigma)$, but $f^{-1}(\{2, 3\}) = \{b, c\}$ is not $\gamma$-$P_S$-closed set in $(X, \tau)$.

**Theorem 5.5.** Let $(X, \tau)$ be $\gamma$-$P_S$-$T_{1\frac{1}{2}}$ space and $\gamma$ be an operation on $\tau$. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\gamma$-$P_S$-continuous if and only if $f$ is $\gamma$-$P_S$-$g$-continuous.

Proof. Follows from Remark 4.2.

**Theorem 5.6.** Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function and $\gamma$ be an operation on $\tau$. If $(X, \tau)$ is $\gamma$-locally indiscrete space, then $f$ is $\gamma$-$P_S$-$g$-continuous.
Proof. This is an immediate consequence of Theorem 3.32.

\[ \Box \]

**Theorem 5.7.** Let \( \gamma \) be an operation on the topological space \((X, \tau)\). If the functions \( f: (X, \tau) \to (Y, \sigma) \) is \( \gamma \)-\( P_S \)-g-continuous and \( g: (Y, \sigma) \to (Z, \rho) \) is continuous. Then the composition function \( g \circ f: (X, \tau) \to (Z, \rho) \) is \( \gamma \)-\( P_S \)-g-continuous.

Proof. It is clear.

**Proposition 5.8.** Let \( \gamma \) be an operation on the topological space \((X, \tau)\). If \( f: (X, \tau) \to (Y, \sigma) \) is a function, \( g: (Y, \sigma) \to (Z, \rho) \) is closed and injective, and \( g \circ f: (X, \tau) \to (Z, \rho) \) is \( \gamma \)-\( P_S \)-g-continuous. Then \( f \) is \( \gamma \)-\( P_S \)-g-continuous.

Proof. Let \( F \) be a closed subset of \( Y \). Since \( g \) is closed, \( g(F) \) is closed subset of \( Z \). Since \( g \circ f \) is \( \gamma \)-\( P_S \)-g-continuous and \( g \) is injective, then \( f^{-1}(F) = f^{-1}(g^{-1}(g(F))) = (g \circ f)(g(F)) \) is \( \gamma \)-\( P_S \)-g-closed in \( X \), which proves that \( f \) is \( \gamma \)-\( P_S \)-g-continuous.

**Definition 5.9.** Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and \( \beta \) be an operation on \( \sigma \). A function \( f: (X, \tau) \to (Y, \sigma) \) is called \( \beta \)-\( P_S \)-g-closed if for every closed set \( F \) of \( X \), \( f(F) \) is \( \beta \)-\( P_S \)-g-closed set in \( Y \).

**Remark 5.10.** Every \( \beta \)-\( P_S \)-closed function is \( \beta \)-\( P_S \)-g-closed.

The converse of the above remark does not true as seen from the following example.

**Example 5.11.** Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\phi, \{c\}, \{b, c\}, X\} \) and \( \sigma = \{\phi, X, \{b\}, \{a, c\}\} \). Define an operation \( \beta \) on \( \sigma \) by \( \beta(A) = A \) for all \( A \in \sigma \). Define \( f: (X, \tau) \to (X, \sigma) \) by \( f(a) = a, f(b) = c \) and \( f(c) = b \). Then \( f \) is \( \beta \)-\( P_S \)-g-closed, but \( f \) is not \( \beta \)-\( P_S \)-closed function since \( \{a\} \) is closed set in \( (X, \tau) \), but \( f(\{a\}) = \{a\} \) is not \( \beta \)-\( P_S \)-closed set in \( (X, \sigma) \).

**Theorem 5.12.** Let \((Y, \sigma)\) be a topological space and \( \beta \) be an operation on \( \sigma \). A function \( f: (X, \tau) \to (Y, \sigma) \) is \( \beta \)-\( P_S \)-g-closed if and only if for each subset \( S \) of \( Y \) and each open set \( O \) in \( X \) containing \( f^{-1}(S) \), there exists a \( \beta \)-\( P_S \)-g-open set \( R \) in \( Y \) such that \( S \subseteq R \) and \( f^{-1}(R) \subseteq O \).

Proof. Suppose that \( f \) is \( \beta \)-\( P_S \)-g-closed function and let \( O \) be an open set in \( X \) containing \( f^{-1}(S) \), where \( S \) is any subset in \( Y \). Then \( f(X \setminus O) \) is \( \beta \)-\( P_S \)-g-open set in \( Y \). If we put \( R = Y \setminus f(X \setminus O) \). Then \( R \) is \( \beta \)-\( P_S \)-g-closed set in \( Y \) containing \( S \) such that \( f^{-1}(R) \subseteq O \).

Conversely, let \( F \) be closed set in \( X \). Let \( S = Y \setminus f(F) \subseteq Y \). Then \( f^{-1}(S) \subseteq X \setminus F \) and \( X \setminus F \) is open set in \( X \). By hypothesis, there exists a \( \beta \)-\( P_S \)-g-open set
R in Y such that $S = Y\setminus f(F) \subseteq R$ and $f^{-1}(R) \subseteq X\setminus F$. For $f^{-1}(R) \subseteq X\setminus F$ implies $R \subseteq f(X\setminus F) \subseteq Y\setminus f(F)$. Hence $R = Y\setminus f(F)$. Since $R$ is $\beta$-$PS$-$g$-open set in Y. Then $f(F)$ is $\beta$-$PS$-$g$-closed set in Y. Therefore, $f$ is $\beta$-$PS$-$g$-closed function.

**Theorem 5.13.** Let $\beta$ be an operation on $\sigma$ and $f: (X, \tau) \to (Y, \sigma)$ be $\beta$-$PS$-$T_{\frac{1}{2}}$ space. Then $f$ is $\beta$-$PS$-$g$-closed if and only if $f$ is $\beta$-$PS$-closed.

**Proof.** Follows from Remark 4.2.

**Theorem 5.14.** Let $f: (X, \tau) \to (Y, \sigma)$ be a function and $\beta$ be an operation on $\sigma$. If $(Y, \sigma)$ is $\beta$-locally indiscrete space, then $f$ is $\beta$-$PS$-$g$-closed.

**Proof.** This is an immediate consequence of Theorem 3.32.

**Definition 5.15.** Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\beta$ be an operation on $\sigma$. A function $f: (X, \tau) \to (Y, \sigma)$ is called $\beta$-$PS$-$g$-open if for every open set $V$ of $X$, $f(V)$ is $\beta$-$PS$-$g$-open set in $Y$.

**Definition 5.16.** A function $f: (X, \tau) \to (Y, \sigma)$ is said to be $\gamma$-$PS$-$g$-homeomorphism, if $f$ is bijective, $\gamma$-$PS$-$g$-continuous and $f^{-1}$ is $\gamma$-$PS$-$g$-continuous.

**Theorem 5.17.** The following statements are equivalent for a bijective function $f: (X, \tau) \to (Y, \sigma)$ with an operation $\beta$ on $\sigma$.

1. $f$ is $\beta$-$PS$-$g$-closed.
2. $f$ is $\beta$-$PS$-$g$-open.
3. $f^{-1}$ is $\beta$-$PS$-$g$-continuous.

**Proof.** It is clear.

**Proposition 5.18.** Let $\alpha$ be an operation on the topological space $(Z, \rho)$. If the function $f: (X, \tau) \to (Y, \sigma)$ is closed (resp., open) and $g: (Y, \sigma) \to (Z, \rho)$ is $\alpha$-$PS$-$g$-closed (resp., $\alpha$-$PS$-$g$-open). Then the composition function $g \circ f: (X, \tau) \to (Z, \rho)$ is $\alpha$-$PS$-$g$-closed (resp., $\alpha$-$PS$-$g$-open).

**Proof.** Obvious.

**Proposition 5.19.** Let $\beta$ be an operation on the topological space $(Y, \sigma)$. If $g: (Y, \sigma) \to (Z, \rho)$ is a function, $f: (X, \tau) \to (Y, \sigma)$ is $\beta$-$PS$-$g$-open and surjective, and $g \circ f: (X, \tau) \to (Z, \rho)$ is continuous. Then $g$ is $\gamma$-$PS$-$g$-continuous.

**Proof.** Similar to Proposition 5.8.
Proposition 5.20. Let $\beta$ be an operation on the topological space $(Y, \sigma)$. If $g: (Y, \sigma) \to (Z, \rho)$ is a function, $f: (X, \tau) \to (Y, \sigma)$ is continuous and surjective, and $g \circ f: (X, \tau) \to (Z, \rho)$ is $\beta$-$P_S$-closed. Then $g$ is $\beta$-$P_S$-g-closed.

Proof. Similar to Proposition 5.8.

References


