Int. Journal of Math. Analysis, Vol. 8, 2014, no. 6, 285 - 300
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2014.4121

γ-P_S-Functions in Topological Spaces

Baravan A. Asaad*, Nazihah Ahmad and Zurni Omar

School of Quantitative Sciences
College of Arts and Sciences
University Utara Malaysia
06010 Sintok, Kedah, Malaysia
* Corresponding author

Copyright © 2014 Baravan A. Asaad, Nazihah Ahmad and Zurni Omar. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

This paper introduces some new classes of functions called γ-P_S-continuous, β-P_S-open and β-P_S-closed using γ-P_S-open set and γ-P_S-closed set. In addition, some properties and characterizations of these functions are given. The result shows that γ-P_S-continuous function and γ-continuous function are independent.

Mathematics Subject Classification: 54A05, 54C05, 54C10

Keywords: γ-P_S-open set, γ-P_S-closed set, γ-P_S-continuous function, β-P_S-open function and β-P_S-closed function

1 Introduction

Kasahara [5] defined the concept of α-closed graphs of an operation γ on τ. Later, Ogata [12] renamed the operation α as γ operation on τ. He defined and investigated the concept of operation-open sets, that is, γ-open sets. Further study by Krishnan and Balachandran ([8], [9]) defined two types of sets called γ-preopen and γ-semiopen sets. Recently, Asaad, Ahmad and Omar [1] defined the notion of γ-regular-open sets which lies strictly between the classes of γ-open set and γ-clopen set. They also introduced a new class of sets called γ-P_S-open sets, and they also defined γ-P_S-operations and their properties [2].

They proved that the union of any class of γ-P_S-open sets in a space X is also a γ-P_S-open, but the intersection of any two γ-P_S-open sets may not be a γ-P_S-open.

In this paper, we define the concept of γ-P_S-continuous function and then establish its properties. The result reveals that γ-P_S-continuous function and γ-continuous function are independent. Furthermore two other classes of functions called β-P_S-open and β-P_S-closed are defined. Some properties and theorems for these two functions are also presented.

Throughout this paper, the pairs (X, τ) and (Y, σ) (or simply X and Y) represent denote topological spaces with no separation axioms assumed unless explicitly stated. Let A be any subset of X, $Int(A)$ and $Cl(A)$ denotes the interior of A and the closure of A, respectively.

2 Preliminaries

A subset A of X is said to be preopen if $A \subseteq Int(Cl(A))$ [11] and semiopen if $A \subseteq Cl(Int(A))$ [10]. The complement of a semiopen set is said to be semiclosed. A preopen subset A of a topological space (X, τ) is said to be P_S-open if for each $x \in A$, there exists a semiclosed set F such that $x \in F \subseteq A$ [6]. An operation γ on the topology τ on X is a mapping $\gamma\colon \tau \rightarrow P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U [12]. A nonempty set A of X with an operation γ on τ is said to be γ-open [12] if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. The complement of a γ-open set is called a γ-closed. The τ_γ-closure of a subset A of X with an operation γ on τ is defined as the intersection of all γ-closed sets containing A and it is denoted by τ_γ-$Cl(A)$ [12], and the τ_γ-interior of a subset A of X with an operation γ on τ is defined as the union of all γ-open sets containing A [9].

A topological space (X, τ) with an operation γ on τ is said to be γ-regular if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U) \subseteq V$ [5]. A topoplogical space (X, τ) with an operation γ on τ is said to be γ-locally indiscrete if every γ-open subset of X is γ-closed, or every γ-closed subset of X is γ-open [1]. A topological space (X, τ) with an operation γ on τ is said to be γ-semi-T_1 if for each pair of distinct points x, y in X, there exist two γ-semiopen sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$ [9]. Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ. A function $f\colon (X, \tau) \rightarrow (Y, \sigma)$ is called γ-continuous if $f^{-1}(V)$ is γ-open set in X, for every open set V in Y [3].

Now we recall some definitions and results which will be used in the sequel.

Remark 2.1 *For any subset A of a topological space (X, τ). Then:*

1. *A is γ-open if and only if τ_γ-$Int(A) = A$ [7].*

2. *A is γ-closed if and only if τ_γ-$Cl(A) = A$ [12].*

Definition 2.2 *Let (X, τ) be a topological space and γ be an operation on τ. A subset A of X is said to be:*

1. *γ-regular-open if $A = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))$ [1].*

2. *γ-preopen if $A \subseteq \tau_\gamma$-$Int(\tau_\gamma$-$Cl(A))$ [8].*

3. *γ-semiopen if $A \subseteq \tau_\gamma$-$Cl(\tau_\gamma$-$Int(A))$ [9].*

4. *γ-β-open if $A \subseteq \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(A)))$ [3]].*

Definition 2.3 *The complement of γ-regular-open, γ-preopen and γ-semiopen set is said to be γ-regular-closed [3], γ-preclosed [8] and γ-semiclosed [9], respectively.*

Definition 2.4 *[2] A γ-preopen subset A of a topological space (X, τ) is called γ-P_S-open if for each $x \in A$, there exists a γ-semiclosed set F such that $x \in F \subseteq A$. The complement of a γ-P_S-open set is called a γ-P_S-closed.*

The family of all γ-P_S-open and γ-preopen subsets of a topological space (X, τ) are denoted by τ_γ-$P_SO(X)$ and τ_γ-$PO(X)$, respectively.

Lemma 2.5 *[2] Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then A is γ-P_S-open if and only if A is γ-preopen set and A is a union of γ-semiclosed sets.*

Definition 2.6 *[2] Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then*

1. *the τ_γ-P_S-interior of A is defined as the union of all γ-P_S-open sets of X contained in A and it is denoted by τ_γ-$P_SInt(A)$.*

2. *the τ_γ-P_S-closure of A is defined as the intersection of all γ-P_S-closed sets of X containing A and it is denoted by τ_γ-$P_SCl(A)$*

Definition 2.7 *[2] A subset N of a topological space (X, τ) is called a γ-P_S-neighbourhood of a point $x \in X$, if there exists a γ-P_S-open set U in X such that $x \in U \subseteq N$.*

Definition 2.8 *[2] Let (X, τ) be a topological space and γ be an operation on τ. Let A be any subset of X. Then*

1. *the γ-P_S-derived set of A is defined as $\{x :$ for every γ-open set U containing x, $U \cap A \setminus \{x\} \neq \phi\}$ and it is denoted by τ_γ-$P_SD(A)$.*

2. *the γ-P_S-boundary of A is defined as τ_γ-$P_SCl(A) \setminus \tau_\gamma$-$P_SInt(A)$ and it is denoted by τ_γ-$P_SBd(A)$.*

Theorem 2.9 *[2] Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then $x \in \tau_\gamma$-$P_SCl(A)$ if and only if $A \cap U \neq \phi$ for every γ-P_S-open set U of X containing x.*

Theorem 2.10 *[2] Let (X, τ) be a topological space and γ be an operation on τ. For any subset A of a space X. The following statements are true.*

1. *A is γ-P_S-open set if and only if τ_γ-$P_SInt(A) = A$.*

2. *A is γ-P_S-closed set if and only if τ_γ-$P_SCl(A) = A$.*

3. *τ_γ-$P_SCl(X \setminus A) = X \setminus \tau_\gamma$-$P_SInt(A)$ and τ_γ-$P_SInt(X \setminus A) = X \setminus \tau_\gamma$-$P_SCl(A)$.*

4. *τ_γ-$P_SD(A) \subseteq \tau_\gamma$-$P_SCl(A)$.*

5. *τ_γ-$P_SCl(A) = \tau_\gamma$-$P_SInt(A) \cup \tau_\gamma$-$P_SBd(A)$.*

Remark 2.11 *If a topological space (X, τ) is γ-regular, then $\tau_\gamma = \tau$ [12] and hence τ_γ-$Int(A) = Int(A)$ [7].*

Theorem 2.12 *[2] Let (X, τ) be a topological space and γ be an operation on τ. Then:*

1. *If X is γ-regular, then the concept of γ-P_S-open set and P_S-open set are equivalent.*

2. *If X is γ-semi-T_1, then the concept of γ-P_S-open set and γ-preopen set are equivalent.*

3. *If X is γ-locally indiscrete, then the concept of γ-P_S-open set and γ-open set are equivalent.*

Definition 2.13 *[6] A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be P_S-continuous if the inverse image of each open set in Y is P_S-open in X.*

Definition 2.14 *[4] Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ. A function $f: (X, \tau) \to (Y, \sigma)$ is called γ-precontinuous at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a γ-preopen set U of X containing x such that $f(U) \subseteq V$. If f is γ-precontinuous at each point x of X, then f is said to be γ-precontinuous.*

Theorem 2.15 *[4] Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ. A function $f: (X, \tau) \to (Y, \sigma)$ is γ-precontinuous if for every open set V in Y, $f^{-1}(V)$ is γ-preopen set in X.*

Definition 2.16 *[4] Let β be an operation on a topological space (Y, σ). A function $f: (X, \tau) \to (Y, \sigma)$ is called β-preopen and β-preclosed if for every open and closed set V of X, $f(V)$ is β-preopen and β-preclosed set in Y, respectively.*

3 γ-P_S-Continuous Functions

In this section, we introduce a new class of functions called γ-P_S-continuous using γ-P_S-open set. Moreover, we give some characterizations and theorems of this function. The result shows that γ-P_S-continuous and γ-continuous functions are independent.

Definition 3.1 *Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ. A function $f: (X, \tau) \to (Y, \sigma)$ is called γ-P_S-continuous at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. If f is γ-P_S-continuous at every point x in X, then f is said to be γ-P_S-continuous.*

Theorem 3.2 *For a function $f: (X, \tau) \to (Y, \sigma)$ and γ be an operation on τ, the following statements are equivalent:*

1. *f is γ-P_S-continuous.*
2. *$f^{-1}(V)$ is γ-P_S-open set in X, for every open set V in Y.*
3. *$f^{-1}(F)$ is γ-P_S-closed set in X, for every closed set F in Y.*
4. *$f(\tau_\gamma\text{-}P_SCl(A)) \subseteq Cl(f(A))$, for every subset A of X.*
5. *$\tau_\gamma\text{-}P_SCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for every subset B of Y.*
6. *$f^{-1}(Int(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$, for every subset B of Y.*
7. *$Int(f(A)) \subseteq f(\tau_\gamma\text{-}P_SInt(A))$, for every subset A of X.*

Proof: (1) $\Rightarrow$ (2) Let V be any open set in Y. We have to show that $f^{-1}(V)$ is γ-P_S-open set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By (1), there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. Which implies that $x \in U \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is γ-P_S-open set in X.

(2) $\Rightarrow$ (3). Let F be any closed set of Y. Then $Y \setminus F$ is an open set of Y. By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is γ-P_S-open set in X and hence $f^{-1}(F)$ is γ-P_S-closed set in X.

(3) $\Rightarrow$ (4). Let A be any subset of X. Then $f(A) \subseteq Cl(f(A))$ and hence $A \subseteq f^{-1}(Cl(f(A)))$. Since $Cl(f(A))$ is closed set in Y. Then by (3), we have $f^{-1}(Cl(f(A)))$ is γ-P_S-closed set in X. Therefore, τ_γ-$P_SCl(A) \subseteq f^{-1}(Cl(f(A)))$. Hence $f(\tau_\gamma\text{-}P_SCl(A)) \subseteq Cl(f(A))$.

(4) $\Rightarrow$ (5). Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (4), we have $f(\tau_\gamma\text{-}P_SCl(f^{-1}(B))) \subseteq Cl(f(f^{-1}(B))) = Cl(B)$. Hence τ_γ-$P_SCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

(5) $\Leftrightarrow$ (6). Let B be any subset of Y. Then apply (5) to $Y \setminus B$ we obtain $\tau_\gamma\text{-}P_SCl(f^{-1}(Y \setminus B)) \subseteq f^{-1}(Cl(Y \setminus B)) \Leftrightarrow \tau_\gamma\text{-}P_SCl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus Int(B)) \Leftrightarrow X \setminus \tau_\gamma\text{-}P_SInt(f^{-1}(B)) \subseteq X \setminus f^{-1}(Int(B)) \Leftrightarrow f^{-1}(Int(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$. Therefore, $f^{-1}(Int(B)) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(B))$.

(6) $\Rightarrow$ (7). Let A be any subset of X. Then $f(A)$ is a subset of Y. By (6), we have $f^{-1}(Int(f(A))) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(f(A))) = \tau_\gamma\text{-}P_SInt(A)$. Therefore, $Int(f(A)) \subseteq f(\tau_\gamma\text{-}P_SInt(A))$.

(7) $\Rightarrow$ (1). Let $x \in X$ and let V be any open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. By (7), we have $Int(f(f^{-1}(V))) \subseteq f(\tau_\gamma\text{-}P_SInt(f^{-1}(V)))$. Then $Int(V) \subseteq f(\tau_\gamma\text{-}P_SInt(f^{-1}(V)))$. Since V is an open set. Then $V \subseteq f(\tau_\gamma\text{-}P_SInt(f^{-1}(V)))$ implies that $f^{-1}(V) \subseteq \tau_\gamma\text{-}P_SInt(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is γ-P_S-open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is γ-P_S-continuous function.

Theorem 3.3 *Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function and γ be an operation on τ. Then f is γ-P_S-continuous if and only if $\tau_\gamma\text{-}P_SBd(f^{-1}(B)) \subseteq f^{-1}(Bd(B))$, for each subset B of Y.*

Proof: Let B be any subset of Y and f be a γ-P_S-continuous function. Then by using Theorem 3.2 (2) and (5), we have $f^{-1}(Bd(B)) = f^{-1}(Cl(B) \setminus Int(B)) = f^{-1}(Cl(B)) \setminus f^{-1}(Int(B)) \supseteq \tau_\gamma\text{-}P_SCl(f^{-1}(B)) \setminus f^{-1}(Int(B)) = \tau_\gamma\text{-}P_SCl(f^{-1}(B)) \setminus \tau_\gamma\text{-}P_SInt(f^{-1}(Int(B))) \supseteq \tau_\gamma\text{-}P_SCl(f^{-1}(B)) \setminus \tau_\gamma\text{-}P_SInt(f^{-1}(B)) = \tau_\gamma\text{-}P_SBd(f^{-1}(B))$. Hence $\tau_\gamma\text{-}P_SBd(f^{-1}(B)) \subseteq f^{-1}(Bd(B))$.

Conversely, let G be any open set in Y. Then $Y \setminus G$ is closed in Y. So by hypothesis, we have $\tau_\gamma\text{-}P_SBd(f^{-1}(Y \setminus G)) \subseteq f^{-1}(Bd(Y \setminus G)) \subseteq f^{-1}(Cl(Y \setminus G)) = f^{-1}(Y \setminus G)$. By Theorem 2.10 (5), $\tau_\gamma\text{-}P_SCl(f^{-1}(Y \setminus G)) = \tau_\gamma\text{-}P_SInt(f^{-1}(Y \setminus G)) \cup \tau_\gamma\text{-}P_SBd(f^{-1}(Y \setminus G)) \subseteq f^{-1}(Y \setminus G)$. Then $f^{-1}(Y \setminus G)$ is γ-P_S-closed set in X and hence $f^{-1}(G)$ is γ-P_S-open set in X. By Theorem 3.2, f is γ-P_S-continuous function.

Theorem 3.4 *Let $f\colon (X,\tau) \to (Y,\sigma)$ be any function and γ be an operation on τ. Then f is γ-P_S-continuous if and only if $f(\tau_\gamma\text{-}P_SD(A)) \subseteq Cl(f(A))$, for each subset A of X.*

Proof: Let f be a γ-P_S-continuous function and A be any subset of X. Then by Theorem 3.2 (4), we have $f(\tau_\gamma\text{-}P_SCl(A)) \subseteq Cl(f(A))$ and by Theorem 2.10 (4), $f(\tau_\gamma\text{-}P_SD(A)) \subseteq f(\tau_\gamma\text{-}P_SCl(A))$ which implies that $f(\tau_\gamma\text{-}P_SD(A)) \subseteq Cl(f(A))$.

Conversely, let F be any closed set in Y. Then $f^{-1}(F)$ is subset of X. By hypothesis, we have $f(\tau_\gamma\text{-}P_SD(f^{-1}(F))) \subseteq Cl(f(f^{-1}(F))) = Cl(F) = F$ and hence $\tau_\gamma\text{-}P_SD(f^{-1}(F)) \subseteq f^{-1}(F)$. Then $f^{-1}(F)$ is γ-P_S-closed set in X. Therefore, by Theorem 3.2, f is γ-P_S-continuous function.

Theorem 3.5 *Let γ be an operation on (X,τ). A function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-P_S-continuous if and only if for every $x \in X$ and for neighbourhood O of Y such that $f(x) \in O$, there exists a γ-P_S-neighbourhood P of X such that $x \in P$ and $f(P) \subseteq O$.*

Proof: It is clear and hence it is omitted.

Theorem 3.6 *Let γ be an operation on (X,τ). A function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-P_S-continuous if and only if the inverse image of every neighbourhood of $f(x)$ is γ-P_S-neighbourhood of $x \in X$.*

Proof: The proof follows from Theorem 3.5.

Theorem 3.7 *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a surjection function and γ be an operation on τ, then the following statements are equivalent:*

1. *f is γ-P_S-continuous.*
2. *$f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(B)))$ and $f^{-1}(Int(B)) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X, for every subset B in Y.*
3. *$Cl(Int(f^{-1}(B))) \subseteq f^{-1}(Cl(B))$ and $f^{-1}(Cl(B)) = \cup_{i\in I}G_i$ where G_i is γ-semiopen set in X, for every subset B in Y.*
4. *$f(Cl(Int(A))) \subseteq Cl(f(A))$ and $f^{-1}(Cl(f(A))) = \cup_{i\in I}G_i$ where G_i is γ-semiopen set in X, for every subset A in X.*
5. *$Int(f(A)) \subseteq f(Int(Cl(A)))$ and $f^{-1}(Int(f(A))) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X, for every subset A in X.*

Proof: It is enough to proof (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1) since the others are obvious.

(1) $\Rightarrow$ (2). Let B be any subset in Y. Then $Int(B)$ is open set in Y. Since f is γ-P_S-continuous, then by Theorem 3.2, $f^{-1}(Int(B))$ is γ-P_S-open set in X. By Lemma 2.5, we obtain $f^{-1}(Int(B))$ is γ-preopen set in X and $f^{-1}(Int(B)) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X, for every subset B in Y. Therefore, $f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(Int(B))))$ and $f^{-1}(Int(B)) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X. Thus $f^{-1}(Int(B)) \subseteq Int(Cl(f^{-1}(B)))$ and $f^{-1}(Int(B)) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X.

(5) $\Rightarrow$ (1). Let V be any open set in Y. Then $f^{-1}(V)$ is a subset of X. By (5), we get $Int(f(f^{-1}(V))) \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(Int(f(f^{-1}(V)))) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X. Hence $Int(V) \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(Int(V)) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X. This implies that $V \subseteq f(Int(Cl(f^{-1}(V))))$ and $f^{-1}(V) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X and hence $f^{-1}(V) \subseteq Int(Cl(f^{-1}(V)))$ and $f^{-1}(V) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X. So $f^{-1}(V)$ is γ-preopen set in X and $f^{-1}(V) = \cup_{i\in I}F_i$ where F_i is γ-semiclosed set in X. Therefore, by Lemma 2.5, $f^{-1}(V)$ is γ-P_S-open set in X and hence by Theorem 3.2, f is γ-P_S-continuous.

Remark 3.8 *Every γ-P_S-continuous function is γ-precontinuous, but the converse is not true as it is shown in the following example.*

Example 3.9 *Let $X = \{a, b, c\}$ with the topologies $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, X\}$. Define an operation $\gamma\colon \tau \to P(X)$ as follows: for every $A \in \tau$*

$$\gamma(A) = \begin{cases} A & \text{if } a \in A \\ Cl(A) & \text{if } a \notin A \end{cases}$$

Then $\tau_\gamma = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}$, τ_γ-$PO(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and τ_γ-$P_SO(X) = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$. Let $f\colon (X, \tau) \to (X, \sigma)$ be a function defined as follows: $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then f is γ-precontinuous, but it is not γ-P_S-continuous since $\{a\} \in \sigma$, but $f^{-1}(\{a\}) = \{b\} \notin \tau_\gamma$-$P_SO(X)$.

Theorem 3.10 *Let (X, τ) be γ-semi-T_1 space and γ be an operation on τ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is γ-P_S-continuous if and only if f is γ-precontinuous.*

Proof: This is an immediate consequence of Theorem 2.12 (2).

Theorem 3.11 *A function $f\colon (X, \tau) \to (Y, \sigma)$ with an operation γ on τ is γ-P_S-continuous if and only if f is γ-precontinuous and for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ-semiclosed set F in X containing x such that $f(F) \subseteq V$.*

Proof: Let $x \in X$ and let V be any open set of Y containing $f(x)$. Since f is γ-P_S-continuous, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. Since U is γ-P_S-open set. Then for each $x \in U$, there exists a γ-semiclosed set F of X such that $x \in F \subseteq U$. Therefore, we get $f(F) \subseteq V$. And also since f is γ-P_S-continuous. Then f is γ-precontinuous.

Conversely, let V be any open set of Y. We have to show that $f^{-1}(V)$ is γ-P_S-open set in X. Since f is γ-precontinuous, then by Theorem 2.15, $f^{-1}(V)$ is γ-preopen set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, there exists a γ-semiclosed set F of X containing x such that $f(F) \subseteq V$. Which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, by Definition 2.4, $f^{-1}(V)$ is γ-P_S-open set in X. Hence by Theorem 3.2, f is γ-P_S-continuous.

Theorem 3.12 *If a function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation γ on τ is γ-P_S-continuous, then for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ-semiclosed set F in X such that $x \in F$ and $f(F) \subseteq V$.*

Proof: Suppose f be a γ-P_S-continuous function and let V be any open set of Y such that $f(x) \in V$, for each $x \in X$. Then there exists a γ-P_S-open set U of X such that $x \in U$ and $f(U) \subseteq V$. Since U is γ-P_S-open set. Then for each $x \in U$, there exists a γ-semiclosed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$. This completes the proof.

Theorem 3.13 *Let (X,τ) be γ-regular space and γ be an operation on τ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-P_S-continuous if and only if f is P_S-continuous.*

Proof: This is an immediate consequence of Theorem 2.12 (1).

The following example shows that the relation between γ-P_S-continuous function and γ-continuous function are independent in general.

Example 3.14 *Let (X,τ) be a topological space and γ be an operation on τ as in Example 3.9. Suppose that $Y = \{1,2,3\}$ and $\sigma = \{\phi, Y, \{2\}, \{2,3\}\}$ be a topology on Y. Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function defined as follows: $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$. Then f is γ-continuous, but it is not γ-P_S-continuous since $\{2,3\} \in \sigma$, but $f^{-1}(\{2,3\}) = \{a,b\} \notin \tau_\gamma$-$P_SO(X)$.*

Example 3.15 *Let (X,τ) be a topological space and γ be an operation on τ as in Example 3.9. Suppose that $Y = \{1,2,3\}$ and $\sigma = \{\phi, Y, \{1,3\}\}$ be a topology on Y. Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function defined as follows: $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. Then f is γ-P_S-continuous, but it is not γ-continuous since $\{1,3\} \in \sigma$, but $f^{-1}(\{1,3\}) = \{a,c\} \notin \tau_\gamma$.*

Theorem 3.16 *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function and let (X,τ) be γ-locally indiscrete topological spaces and γ be an operation on τ. Then f is γ-P_S-continuous if and only if f is γ-continuous.*

Proof: This is an immediate consequence of Theorem 2.12 (3).

Theorem 3.17 *For any operation γ on τ and $f\colon (X,\tau) \to (Y,\sigma)$ be any function, then: $X \setminus \tau_\gamma$-$P_SC(f) = \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$, where τ_γ-$P_SC(f)$ denotes the set of points at which f is γ-P_S-continuous function.*

Proof: Let $x \in \tau_\gamma$-$P_SC(f)$. Then there exists open set V in (Y,σ) containing $f(x)$ such that $f(U) \nsubseteq V$ for every γ-P_S-open set U of (X,τ) containing x. Hence $U \cap X \setminus f^{-1}(V) \neq \phi$ for every γ-P_S-open set U of (X,τ) containing x. Therefore, by Theorem 2.9, $x \in \tau_\gamma$-$P_SCl(X \setminus f^{-1}(V))$. Then $x \in f^{-1}(V) \cap \tau_\gamma$-$P_SCl(X \setminus f^{-1}(V)) \subseteq \tau_\gamma$-$P_SCl(f^{-1}(V)) \cap \tau_\gamma$-$P_SCl(X \setminus f^{-1}(V)) = \tau_\gamma$-$P_SBd(f^{-1}(V))$. Then $X \setminus \tau_\gamma$-$P_SC(f) \subseteq \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$.

Conversely, let $x \notin X \setminus \tau_\gamma$-$P_SC(f)$. Then for each open set V in (Y,σ) containing $f(x)$, $f^{-1}(V)$ is γ-P_S-open set of (X,τ) containing x. Hence $x \in \tau_\gamma$-$P_SInt(f^{-1}(V))$ and hence $x \notin \tau_\gamma$-$P_SBd(f^{-1}(V))$ for every open set V in (Y,σ) containing $f(x)$. Therefore, $X \setminus \tau_\gamma$-$P_SC(f) \supseteq \cup\{\tau_\gamma$-$P_SBd(f^{-1}(V)) : V$ is open in (Y,σ) such that $f(x) \in V$ for each $x \in X\}$.

Theorem 3.18 *Let γ be an operation on the topological space (X,τ). If the functions $f\colon (X,\tau) \to (Y,\sigma)$ is γ-P_S-continuous and $g\colon (Y,\sigma) \to (Z,\rho)$ is continuous. Then the composition function $g \circ f\colon (X,\tau) \to (Z,\rho)$ is γ-P_S-continuous.*

Proof: Clear.

Proposition 3.19 *Let γ be an operation on the topological space (X,τ). If $f\colon (X,\tau) \to (Y,\sigma)$ is a function, $g\colon (Y,\sigma) \to (Z,\rho)$ is open and injective, and $g \circ f\colon (X,\tau) \to (Z,\rho)$ is γ-P_S-continuous. Then f is γ-P_S-continuous.*

Proof: Let V be an open subset of Y. Since g is open, $g(V)$ is open subset of Z. Since $g \circ f$ is γ-P_S-continuous and g is injective, then $f^{-1}(V) = f^{-1}(g^{-1}(g(V)) = (g \circ f)(g(V))$ is γ-P_S-open in X, which proves that f is γ-P_S-continuous.

Definition 3.20 *A subset A of a topological space (X,τ) with an operation γ on τ is called γ-γ-P_S-open and τ-γ-P_S-open if τ_γ-$Int(A) = \tau_\gamma$-$P_SInt(A)$ and $Int(A) = \tau_\gamma$-$P_SInt(A)$, respectively.*

Lemma 3.21 *Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:*

1. *A is γ-γ-P_S-open and γ-P_S-open.*
2. *A is γ-γ-P_S-open and γ-open.*
3. *A is γ-P_S-open and γ-open.*

Proof: Follows from Definition 3.20, Remark 2.1 (1) and Theorem 2.10 (1).

Lemma 3.22 *Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:*

1. *A is τ-γ-P_S-open and γ-P_S-open.*
2. *A is τ-γ-P_S-open and open.*
3. *A is γ-P_S-open and open.*

Proof: Follows from Definition 3.20 and Theorem 2.10 (1).

Proposition 3.23 *In a γ-regular space (X, τ), then the concept of γ-γ-P_S-open set and τ-γ-P_S-open set are equivalent.*

Proof: The proof follows form Definition 3.20 and Remark 2.11.

Definition 3.24 *Let (X, τ) and (Y, σ) be two topological spaces and γ be an operation on τ. A function $f: (X, \tau) \to (Y, \sigma)$ is called τ-γ-P_S-continuous and γ-γ-P_S-continuous if for each open set V of Y, $f^{-1}(V)$ is τ-γ-P_S-open and γ-γ-P_S-open sets in X, respectively.*

Theorem 3.25 *For a function $f: (X, \tau) \to (Y, \sigma)$ and γ be an operation on τ, the following statements are equivalent:*

1. *f is γ-γ-P_S-continuous and γ-P_S-continuous.*
2. *f is γ-γ-P_S-continuous and γ-continuous.*
3. *f is γ-P_S-continuous and γ-continuous.*

Proof: The proof follows from Lemma 3.21.

Theorem 3.26 *For a function $f: (X, \tau) \to (Y, \sigma)$ and γ be an operation on τ, the following statements are equivalent:*

1. *f is τ-γ-P_S-continuous and γ-P_S-continuous.*

2. *f is τ-γ-P_S-continuous and continuous.*

3. *f is γ-P_S-continuous and continuous.*

Proof: Follows from Lemma 3.22.

Theorem 3.27 *Let (X,τ) be γ-regular space and γ be an operation on τ. Then a function $f\colon (X,\tau) \to (Y,\sigma)$ is γ-γ-P_S-continuous if and only if f is τ-γ-P_S-continuous.*

Proof: This is an immediate consequence of Proposition 3.23.

4 β-P_S-Open and β-P_S-Closed Functions

Definition 4.1 *Let (X,τ) and (Y,σ) be two topological spaces and β be an operation on σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called β-P_S-open if for every open set V of X, $f(V)$ is β-P_S-open set in Y.*

Definition 4.2 *Let (X,τ) and (Y,σ) be two topological spaces and β be an operation on σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ called β-P_S-closed if for every closed set F of X, $f(F)$ is β-P_S-closed set in Y.*

Theorem 4.3 *Let β be an operation on (Y,σ). A function $f\colon (X,\tau) \to (Y,\sigma)$ is β-P_S-open if and only if for every $x \in X$ and for every neighbourhood N of x, there exists a β-P_S-neighbourhood M of Y such that $f(x) \in M$ and $M \subseteq f(N)$.*

Proof: Obvious.

Theorem 4.4 *The following statements are equivalent for a function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation β on σ:*

1. *f is β-P_S-open.*

2. *$f(Int(A)) \subseteq \sigma_\beta$-$P_SInt(f(A))$, for every $A \subseteq X$.*

3. *$Int(f^{-1}(B)) \subseteq f^{-1}(\sigma_\beta$-$P_SInt(B))$, for every $B \subseteq Y$.*

4. *$f^{-1}(\sigma_\beta$-$P_SCl(B)) \subseteq Cl(f^{-1}(B))$, for every $B \subseteq Y$.*

5. *σ_β-$P_SCl(f(A)) \subseteq f(Cl(A))$, for every $A \subseteq X$.*

6. *σ_β-$P_SD(f(A)) \subseteq f(Cl(A))$, for every $A \subseteq X$.*

Proof: The proof is similar to Theorem 3.2.

Remark 4.5 *Every β-P_S-open and β-P_S-closed function is β-preopen and β-preclosed respectively, but the converse is not true as it is shown in the following example.*

Example 4.6 *Let $X = \{a, b, c\}$ with the topologies $\tau = \{\phi, \{c\}, \{b, c\}, X\}$ and $\sigma = \{\phi, X, \{b\}, \{a, c\}\}$. Define an operation β on σ by $\beta(A) = A$ for all $A \in \sigma$. Define $f\colon (X, \tau) \to (X, \sigma)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is both β-preopen and β-preclosed, but f is not β-P_S-open and β-P_S-closed function since $\{b, c\} \in \tau$ and $\{a\}$ is closed set in (X, τ), but $f(\{b, c\}) = \{a, c\}$ is not β-P_S-open set in (X, σ) and $f(\{a\}) = \{a\}$ is not β-P_S-closed set in (X, σ), respectively.*

Theorem 4.7 *Let (Y, σ) be β-semi-T_1 space and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is β-P_S-open if and only if f is β-preopen.*

Proof: Follows from Theorem 2.12 (2).

Definition 4.8 *Let (X, τ) and (Y, σ) be two topological spaces and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is called β-open if for every open set V of X, $f(V)$ is β-open set in Y.*

Theorem 4.9 *Let (Y, σ) be β-locally indiscrete topological spaces and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is β-P_S-open if and only if f is β-open.*

Proof: Follows from Theorem 2.12 (3).

Theorem 4.10 *Let (Y, σ) be a topological space and β be an operation on σ. A function $f\colon (X, \tau) \to (Y, \sigma)$ is β-P_S-closed if and only if for each subset S of Y and each open set O in X containing $f^{-1}(S)$, there exists a β-P_S-open set R in Y containing S such that $f^{-1}(R) \subseteq O$.*

Proof: Suppose that f is β-P_S-closed function and let O be an open set in X containing $f^{-1}(S)$, where S is any subset in Y. Then $f(X \setminus O)$ is β-P_S-open set in Y. If we put $R = Y \setminus f(X \setminus O)$. Then R is β-P_S-closed set in Y such that $S \subseteq R$ and $f^{-1}(R) \subseteq O$.

Conversely, let F be closed set in X. Let $S = Y \setminus f(F) \subseteq Y$. Then $f^{-1}(S) \subseteq X \setminus F$ and $X \setminus F$ is open set in X. By hypothesis, there exists a β-P_S-open set R in Y such that $S = Y \setminus f(F) \subseteq R$ and $f^{-1}(R) \subseteq X \setminus F$. For $f^{-1}(R) \subseteq X \setminus F$ implies $R \subseteq f(X \setminus F) \subseteq Y \setminus f(F)$. Hence $R = Y \setminus f(F)$. Since R is β-P_S-open set in Y. Then $f(F)$ is β-P_S-closed set in Y. Therefore, f is β-P_S-closed function.

Definition 4.11 *A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be γ-P_S-homeomorphism, if f is bijective, γ-P_S-continuous and f^{-1} is γ-P_S-continuous.*

Theorem 4.12 *The following statements are equivalent for a bijective function $f\colon (X,\tau) \to (Y,\sigma)$ with an operation β on σ.*

1. *f is β-P_S-closed.*
2. *f is β-P_S-open.*
3. *f^{-1} is β-P_S-continuous.*

Proof: It is clear.

Proposition 4.13 *Let α be an operation on the topological space (Z,ρ). If the function $f\colon (X,\tau) \to (Y,\sigma)$ is open (resp., closed) and $g\colon (Y,\sigma) \to (Z,\rho)$ is α-P_S-open (resp., α-P_S-closed). Then the composition function $g \circ f\colon (X,\tau) \to (Z,\rho)$ is α-P_S-open (resp., α-P_S-closed).*

Proof: Obvious.

Proposition 4.14 *Let β be an operation on the topological space (Y,σ). If $g\colon (Y,\sigma) \to (Z,\rho)$ is a function, $f\colon (X,\tau) \to (Y,\sigma)$ is β-P_S-open and surjective, and $g \circ f\colon (X,\tau) \to (Z,\rho)$ is continuous. Then g is γ-P_S-continuous.*

Proof: Similar to Proposition 3.19.

Proposition 4.15 *Let β be an operation on the topological space (Y,σ). If $g\colon (Y,\sigma) \to (Z,\rho)$ is a function, $f\colon (X,\tau) \to (Y,\sigma)$ is continuous and surjective, and $g \circ f\colon (X,\tau) \to (Z,\rho)$ is β-P_S-open. Then g is β-P_S-open.*

Proof: Similar to Proposition 3.19.

Definition 4.16 *Let (X,τ) and (Y,σ) be two topological spaces and β be an operation on σ. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called σ-β-P_S-open and β-β-P_S-open if for every open set V of X, $f(V)$ is σ-β-P_S-open and β-β-P_S-open sets in Y, respectively.*

Theorem 4.17 *For a function $f\colon (X,\tau) \to (Y,\sigma)$ and β be an operation on σ, the following statements are equivalent:*

1. *f is β-β-P_S-open (resp., β-β-P_S-closed) and β-P_S-open (resp., β-P_S-closed).*
2. *f is β-β-P_S-open (resp., β-β-P_S-closed) and β-open (resp., β-closed).*
3. *f is β-P_S-open (resp., β-P_S-closed) and β-open (resp., β-closed).*

Proof: Follows from Lemma 3.21.

Theorem 4.18 *For a function $f\colon (X,\tau) \rightarrow (Y,\sigma)$ and β be an operation on σ, the following statements are equivalent:*

1. *f is σ-β-P_S-open (resp., σ-β-P_S-closed) and β-P_S-open (resp., β-P_S-closed).*

2. *f is σ-β-P_S-open (resp., σ-β-P_S-closed) and open (resp., closed).*

3. *f is β-P_S-open (resp., β-P_S-closed) and open (resp., closed).*

Proof: Follows from Lemma 3.22.

Theorem 4.19 *Let (Y,σ) be β-regular space and β be an operation on σ. Then a function $f\colon (X,\tau) \rightarrow (Y,\sigma)$ is β-β-P_S-open (resp., β-β-P_S-closed) if and only if f is σ-β-P_S-open (resp., σ-β-P_S-closed).*

Proof: This is an immediate consequence of Proposition 3.23.

References

[1] B.A. Asaad, Nazihah Ahmad and Zurni Omar, γ-Regular-open sets and γ-extremally disconnected spaces, *Mathematical Theory and Modeling,* **9** *12* (2013), 131-140.

[2] B.A. Asaad, Nazihah Ahmad and Zurni Omar, γ-P_S-open sets in topological spaces, *Proceedings of 1st Innovation and Analytics Conference and Exhibition, UUM Press, Sintok,* (2013). (to be appeared)

[3] C.K. Basu, B.M.U. Afsan, and M.K. Ghosh, A class of functions and separation axioms with respect to an operation, *Hacettepe Journal of Mathematics and Statistics,* **38** *2* (2009), 103-118.

[4] M.K. Ghost, On γ-Preopen Sets, *Int. Journal of Math. Analysis,* **6** *53* (2012), 2633-2645.

[5] S. Kasahara, Operation compact spaces, *Math. Japonica,* **24** *1* (1979), 97-105.

[6] A.B. Khalaf and B.A. Asaad, P_S-open sets and P_S-continuity in topological spaces, *J. Duhok univ.,* **12** *2* (2009), 183-192.

[7] G.S.S. Krishnan, A new class of semi open sets in a topological space, *Proceedings of the Second National Conference on Mathematical and Computational Models, Allied Publishers, New Delhi,* (2003), 305-314.

[8] G.S.S. Krishnan and K. Balachandran, On a class of γ-preopen sets in a topological space, *East Asian Math. J.,* **22** *2* (2006), 131-149.

[9] G.S.S. Krishnan and K. Balachandran, On γ-semiopen sets in topological spaces, *Bull. Cal. Math. Soc.,* **98** *6* (2006), 517-530.

[10] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly,* **70** *1* (1963), 36-41.

[11] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On precontinuous and week precontinuous mappings, *Proc. Math. Phys. Soc. Egypt,* **53** (1982), 47-53.

[12] H. Ogata, Operation on topological spaces and associated topology, *Math. Japonica,* **36** *1* (1991), 175-184.

Received: January 7, 2014