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Improved Time Response of Stabilization in Synchronization of Chaotic Oscillators Using *Mathematica*

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Academic Editor: Ockie Bosch

Received: 27 December 2015; Accepted: 15 June 2016; Published: 22 June 2016

Abstract: Chaotic dynamics are an interesting topic in nonlinear science that has been intensively studied during the last three decades due to its wide availability. Motivated by much researches on synchronization, the authors of this study have improved the time response of stabilization when parametrically excited Φ^6—Van der Pol Oscillator (VDPO) and Φ^6—Duffing Oscillator (DO) are synchronized identically as well as non-identically (with each other) using the Linear Active Control (LAC) technique using *Mathematica*. Furthermore, the authors have synchronized the same pairs of the oscillators using a more robust synchronization with faster time response of stability called Robust Adaptive Sliding Mode Control (RASMC). A comparative study has been done between the previous results of Njah's work and our results based on *Mathematica* via LAC. The time response of stabilization of synchronization using RASMC has been discussed.

Keywords: chaos synchronization; LAC; *Mathematica*; chaotic oscillators; RASMC

1. Introduction

The study of chaotic behavior in nonlinear systems has attracted much attention because of many possible applications in various fields of science and technology. Most of the research has been devoted to the modeling of new chaotic systems together with the control and synchronization [1]. Thus far much work based on modeling, as well as various new control and synchronization techniques, has been carried out and is worth citing. For example, the sliding mode control [2–5], adaptive control [6–8], linear active control [9–13], linear feedback control [14–16], projective synchronization [17–19], nonlinear active control [20,21] and backstepping control [22], to mention but a few.

A recent study of Shahzad [23] focused the attention of researchers on how to choose a model based synchronization technique and the appropriate mathematical tools for simulation. Pourmahmood *et al.* [24] have developed a Robust Adaptive Sliding Mode Control (RASMC) and implemented it successfully on three well known chaotic systems (Lorenz, Chen and Liu) using *MATLAB* for all of the simulations, but when the same study was done using *Mathematica* by Shahzad [23], remarkable changes were observed in terms of time response to stabilize the synchronization.

Motivated by the aforementioned studies, we synchronize the identical and non-identical pairs of Φ^6—VDPO and DO using the LAC and RASMC, respectively. However, the synchronization of Φ^6—VDPO and DO using the LAC has already been done by Njah [12] but when the same work of Njah [12] was repeated via LAC and using *Mathematica*, remarkable changes had been found in the time response of stabilization of synchronization. On the other hand, the faster time response of stabilization of synchronization performances of the RASMC forced us to implement it on the same pairs of systems studied by Njah [12]. To the best of our knowledge, this kind of study has never been done before. The main objective behind the implementation of RASMC is that the sliding mode control is one of the robust control methods and has many interesting features such as low sensitivity to external disturbances and robustness to the plant uncertainties due to structural variations and un-modeled dynamics. The sliding mode controller is composed of an equivalent control part that describes the behavior of the system when the trajectories stay over the sliding surface and a variable structure control part that enforces the trajectories to reach the sliding surface and remain on it evermore. The adaptive control is a suitable approach to overcome system uncertainties, especially uncertainties derived from uncertain parameters. The adaptive sliding mode control has the advantages of combining the robustness of the sliding mode control with the tracking facilities of the adaptive control ([24], and the references therein).

The rest of the paper has been organized as follows: In Section 2, the three identical pairs of Φ^6—VDPO and DO, respectively, and non-identical pairs of Φ^6—VDPO and DO have been synchronized using the LAC. Section 3 is devoted to the brief description of RASMC as well as its implementation on identical synchronization of Φ^6—VDPO and DO, respectively, and non-identical pair of Φ^6—VDPO and DO. Lastly, the whole study has been concluded in Section 4.

2. Synchronization Using LAC

In this section, we synchronize the identical and non-identical pairs of Φ^6—VDPO and DO, respectively, using the LAC technique for chaos synchronization that was proposed by Bai and Lonngren [9] and it has recently been accepted as one of the most efficient techniques for synchronizing both identical and non-identical chaotic systems because of its simple implementation in practical systems [25–28]. It can be easily designed according to the given conditions of the chaotic system as a way of accomplishing synchronization globally asymptotically, if the nonlinearity of the system is known. There are no derivatives in the controller and no need to calculate the Lyapunov exponents to execute the controller. These characteristics give an advantage to the technique over other conventional synchronization techniques.

2.1. Description of the Models

Since the chaotic systems are very complex nonlinear systems, they are exceptionally sensitive to tiny changes in their initial conditions and parameters variations. With the passage of time and due to the potential applications of chaotic systems in certain scientific fields, many chaotic and hyperchaotic systems have been investigated (Lorenz, Chen and Liu, *etc*.). In this direction, the Φ^6—VDPO and DO are periodically self-excited and have rich applications in various disciplines like electronics, physics, engineering, neurology and biological sciences [29–31]. Njah [12] studied and investigated the synchronizations of Φ^6—VDPO and DO with applications to secure communication using the LAC technique based on the Lyapunov stability theory and the Routh-Hurwitz criterion. The synchronization schemes have been studied without considering the external disturbances and model uncertainties. However, in practical applications, either environmental changes (noise) may occur any time or lack of parameters knowledge may disturb the stability of the synchronized system and this uncertainty or environmental noise cannot be simply ignored.

The Φ^6—VDPO and DO are classical examples of self-oscillatory and periodic systems and are now considered as very valuable mathematical models that can be utilized in much more complex and modified systems. In these models, there exist two frequencies, namely periodic forcing and self-oscillations. The energy is generated at low amplitudes and dissipated at high amplitude. The dynamics of chaotic parametrically excited Φ^6—VDPO [12] are given by the following mathematical model:

$$\Phi^6 - \text{VDPO}: \quad \ddot{x} - \mu_1(1-x^2)\dot{x} + \alpha_1\left\{1+\eta_1\cos(2\omega_1 t)\right\}x + \beta_1 x^3 + \lambda_1 x^5 = f_1\cos\omega_1 t \tag{1}$$

The dynamics of another chaotic parametrically excited Φ^6—DO [12] are given by the following mathematical model:

$$\Phi^6 - \mathrm{DO}: \; \ddot{x} + \mu_2\dot{x} + \alpha_2\{1 + \eta_2\cos(2\omega_2 t)\}\,x + \beta_2 x^3 + \lambda_2 x^5 = f_2\cos\omega_2 t \tag{2}$$

where the Φ^6—VDPO and DO exhibit chaotic attractors for the following parameter values: $\mu_1 = 0.4$, $\alpha = 1.0$, $\beta = -0.7$, $\lambda_1 = 0.1$, $\eta_1 = 0.7$, $f_1 = 9$, $\omega_1 = 3.14$ and $\mu_2 = 0.4$, $\alpha_2 = 0.46$, $\beta_2 = 1$, $\lambda_2 = 0.1$, $f_2 = 4.5$, $\eta_2 = 0.7$, $\omega_2 = 0.86$, respectively [12].

2.2. Synchronization of Two Identical Φ^6—VDPO Oscillators via LAC

To achieve synchronization between two identical Φ^6—VDPO, let us consider the master–slave systems synchronization scheme for two coupled identical chaotic Φ^6—VDPO that can be written by choosing $x = x_1$ and $\dot{x}_1 = x_2$ and $x = y_1$ and $\dot{x} = y_2$ in Equation (1) for master and slave systems, respectively, as follows:

$$\text{Master System}: \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \mu_1(1 - x_1^2)x_2 - \alpha_1\{1 + \eta_1\cos(2\omega_1 t)\}\,x_1 - \beta_1 x_1^3 - \lambda_1 x_1^5 + f_1\cos\omega_1 t \end{cases} \tag{3}$$

$$\text{Slave System}: \begin{cases} \dot{y}_1 = y_2 + u_1(t), \\ \dot{y}_2 = \mu_1(1 - y_1^2)y_2 - \alpha_1\{1 + \eta_1\cos(2\omega_1 t)\}\,y_1 - \beta_1 y_1^3 - \lambda_1 y_1^5 + f_1\cos\omega_1 t + u_2(t) \end{cases} \tag{4}$$

where $[x_1(t),\, x_2(t)]^T$ and $[y_1(t),\, y_2(t)]^T \in R^2$ are the state variables of master and slave systems, respectively; μ_1, α_1, β_1, λ_1, f_1, η_1 and ω_1 are the parameters involved in Equations (3) and (4) and $\boldsymbol{u}(t) = [u_1(t), u_2(t)]^T \in \boldsymbol{R}^{2\times 1}$ are the control inputs yet to be determined.

Now, the error dynamics ($e_i = y_i - x_i$, for $i = 1, 2$) from Equations (3) and (4) can be written as follows:

$$\begin{aligned} \dot{e}_1 &= e_2 + u_1(t) \\ \dot{e}_2 &= \mu_1 e_1 - \mu_1(y_1^2 y_2 - x_1^2 x_2) - \alpha_1\{1 + \eta_1\cos(2\omega_1 t)\}\,e_1 - \beta_1(y_1^3 - x_1^3) - \lambda_1(y_1^5 - x_1^5) + u_2(t) \end{aligned} \tag{5}$$

In order to make the error dynamics linear, let us redefine u_1 & u_2 as follows:

$$\begin{aligned} u_1(t) &= v_1(t) \\ u_2(t) &= \mu_1(y_1^2 y_2 - x_1^2 x_2) + \beta_1(y_1^3 - x_1^3) + \lambda_1(y_1^5 - x_1^5) + v_2(t) \end{aligned} \tag{6}$$

$\therefore$ Now the linear error dynamical system can be written as:

$$\begin{aligned} \dot{e}_1 &= e_2 + v_1(t) \\ \dot{e}_2 &= \mu_1 e_1 - \alpha_1\{1 + \eta_1\cos(2\omega_1 t)\}\,e_1 + v_2(t) \end{aligned} \tag{7}$$

The linear error dynamics (Equation (7)) is controlled by $v_1(e_1, e_2)$ and $v_2(e_1, e_2)$ that are defined as: $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = D\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ where $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a constant feedback matrix yet to be determined and the error dynamics (Equation (7)) can be written as: $\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = C\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ where $C = \begin{pmatrix} a & 1 + b \\ c - \alpha_1\{1 + \eta_1\cos(2\omega_1 t)\} & \mu_1 + d \end{pmatrix}$ is the coefficient matrix. According to the Lyapunov stability theory and Routh–Hurwitz criteria, choose $a + d + \mu_1 < 0$ and $\{c - \alpha_1(1 + \eta_1\cos 2\omega_1 t)\}(1 + b) - a(\mu_1 + d) < 0$ for the stabilization of the synchronization of Equations (3) and (4).

Let $a + d + \mu_1 = \{c - \alpha_1(1 + \eta_1 \cos 2\omega_1 t)\}(1 + b) - a(\mu_1 + d) = -E$ for $E > 0$. We choose $a = b = 0$ and $E = 1$, which yields $u_1(t) = 0$ and $u_2(t) = \mu_1(y_1^2 y_2 - x_1^2 x_2) + \beta_1(y_1^3 - x_1^3) + \lambda_1(y_1^5 - x_1^5) + \{\alpha_1(1 + \eta_1 \cos 2\omega_1 t) - E\} e_1 - (\mu_1 + E)e_2$.

For the same values of parameters ($\mu_1 = 0.4$, $\alpha_1 = 1$, $\beta_1 = -0.7$, $\lambda_1 = 0.1$, $f_1 = 9$, $\eta_1 = 0.7$, $\omega_1 = 3.14$ and $E = 1$) and initial conditions ($x_1(0) = 0.1$; $x_2(0) = 0.2$; $y_1(0) = 2.2$; $y_2(0) = 0.05$) taken by [12], we have repeated all simulations using *Mathematica*. The following are the graphs of synchronization of two identical Φ^6—VDPO oscillators:

2.3. Synchronization for Two Identical Φ^6—DO via LAC

To achieve synchronization between two identical Φ^6—DO, let us consider the master–slave systems synchronization scheme for two coupled identical chaotic Φ^6—DO that can be written by taking $x = x_1$ and $\dot{x}_1 = x_2$ and $x = y_1$ and $\dot{x} = y_2$ in Equation (2) for master and slave systems, respectively, as follows:

$$\text{Master System}: \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\mu_2 x_2 - \alpha_2 \{1 + \eta_2 \cos(2\omega_2 t)\} x_1 - \beta_2 x_1^3 - \lambda_2 x_1^5 + f_2 \cos\omega_2 t \end{cases} \tag{8}$$

$$\text{Slave System}: \begin{cases} \dot{y}_1 = y_2 + u_1(t), \\ \dot{y}_2 = -\mu_2 y_2 - \alpha_2 \{1 + \eta_2 \cos(2\omega_2 t)\} y_1 - \beta_2 y_1^3 - \lambda_2 y_1^5 + f_2 \cos\omega_2 t + u_2(t) \end{cases} \tag{9}$$

where $[x_1(t),\ x_2(t)]^T$ and $[y_1(t),\ y_2(t)]^T \in R^2$ are the state variables; μ_2, α_2, β_2, λ_2, f_2, η_2 and ω_2 are the parameters involved in Equations (8) and (9); and $\boldsymbol{u}(t) = [u_1(t), u_2(t)]^T \in \boldsymbol{R}^{2\times 1}$ are the control inputs yet to be determined.

Using LAC technique as in Section 2.2, someone can find the controllers $u_1(t) = 0$ and $u_2(t) = \beta_2(y_1^3 - x_1^3) + \lambda_2(y_1^5 - x_1^5) + \{\alpha_2(1 + \eta_2 \cos 2\omega_2 t) - E\} e_1 + (\mu_2 - E)e_2$. For the same values of parameters ($\mu_2 = 0.4$, $\alpha_2 = 0.46$, $\beta_2 = 1$, $\lambda_2 = 0.1$, $f_2 = 4.5$, $\eta_2 = 0.7$, $\omega_2 = 0.86$ and $E = 1$) and initial conditions ($x_1(0) = 0$; $x_2(0) = 1.5$; $y_1(0) = 0.5$; $y_2(0) = 1$) as taken by Njah [12], we have repeated all simulations using *Mathematica*. The following are the graphs of synchronization of two identical Φ^6—DO:

2.4. Synchronization for Φ^6—VDPO and DO via LAC

To achieve synchronization between Φ^6—VDPO and DO, let us consider the master-slave systems synchronization scheme for two coupled Φ^6—VDPO and DO that can be written by taking $x = x_1$ and $\dot{x}_1 = x_2$ and $x = y_1$ and $\dot{x} = y_2$ in Equations (1) and (2) for master and slave systems, respectively, as follows:

$$\text{Master System}: \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \mu_1(1 - x_1^2)x_2 - \alpha_1 \{1 + \eta_1 \cos(2\omega_1 t)\} x_1 - \beta_1 x_1^3 - \lambda_1 x_1^5 + f_1 \cos\omega_1 t \end{cases} \tag{10}$$

$$\text{Slave System}: \begin{cases} \dot{y}_1 = y_2 + u_1(t), \\ \dot{y}_2 = -\mu_2 y_2 - \alpha_2 \{1 + \eta_2 \cos(2\omega_2 t)\} y_1 - \beta_2 y_1^3 - \lambda_2 y_1^5 + f_2 \cos\omega_2 t + u_2(t) \end{cases} \tag{11}$$

where $[x_1(t),\ x_2(t)]^T$ and $[y_1(t),\ y_2(t)]^T \in R^2$ are the state variables of master and slave systems; μ_1, α_1, β_1, λ_1, f_1, η_1, ω_1, μ_2, α_2, β_2, λ_2, f_2, η_2 and ω_2 are the parameters involved in Equations (10) and (11) and $\boldsymbol{u}(t) = [u_1(t), u_2(t)]^T \in \boldsymbol{R}^{2\times 1}$ are the control inputs yet to be determined.

Using LAC technique, as it has been implemented in the last two subsections, someone can find the controllers:

$$\begin{aligned} u_1(t) &= 0 \\ u_2(t) = \ & (\mu_1 + \mu_2)x_2 - \mu_1 x_1^2 x_2 + \alpha_2(1 + \eta_2 \cos 2\omega_2 t)x_1 - \alpha_1(1 + \eta_1 \cos 2\omega_1 t)x_1 + \beta_2 y_1^3 - \beta_1 x_1^3 \\ & + \lambda_2 y_1^5 - \lambda_1 x_1^5 - f_2 \cos\omega_2 t + f_1 \cos\omega_1 t + \{\alpha_2(1 + \eta_2 \cos 2\omega_2 t) - E\} e_1 + (\mu_2 - E)e_2 \end{aligned} \tag{12}$$

For the same values of parameters ($\mu_1 = 0.4$, $\alpha_1 = 1$, $\beta_1 = -0.7$, $\lambda_1 = 0.1$, $f_1 = 9$, $\eta_1 = 0.7$, $\omega_1 = 3.14$, $\mu_2 = 0.4$, $\alpha_2 = 0.46$, $\beta_2 = 1$, $\lambda_2 = 0.1$, $f_2 = 4.5$, $\eta_2 = 0.7$, $\omega_2 = 0.86$ and $E = 1$) and initial conditions ($x_1(0) = 0.1$; $x_2(0) = 0.2$; $y_1(0) = 0$; $y_2(0) = 1.5$) as taken by Njah [12], we have repeated all simulations using *Mathematica*. The following are the graphs of synchronization of Φ^6—VDPO and DO.

2.5. Results and Discussions

In Sections 2.2–2.4, three different pairs of Φ^6—VDPO and DO have been synchronized using LAC technique. In our study, all simulations are based on *Mathematica* that provide us the remarkable changes in the time of stabilization of synchronization. Earlier, in the same study of Njah [12], for identical pairs of Φ^6—VDPO and non-identical pairs (*i.e.*, Φ^6—VDPO and DO), controllers were activated at around $t = 60$ and for identical pairs of Φ^6—DO, the controllers were activated at around $t = 100$. On the other hand, for the same pairs and same technique (LAC) if *Mathematica* is being used, someone can observe the remarkable changes in the time of stabilization (Figures 1–13). In our study, for all of the cases, not only do controllers activate around $t = 4$ but secure communication scheme (Figures 3, 4, 7, 8, 11 and 12), convergence of errors defined by $e(t) = \sqrt{e_1^2(t) + e_2^2(t)}$ (Figure 13) also start to stabilize at the same time when simulation is done using *Mathematica*. Furthermore, it may also be observed that the error states converged to the origin in the range of $[-0.5, 1.5]$ very smoothly and quickly as compared to the work done by Njah [12]. These features give advantages to the current study.

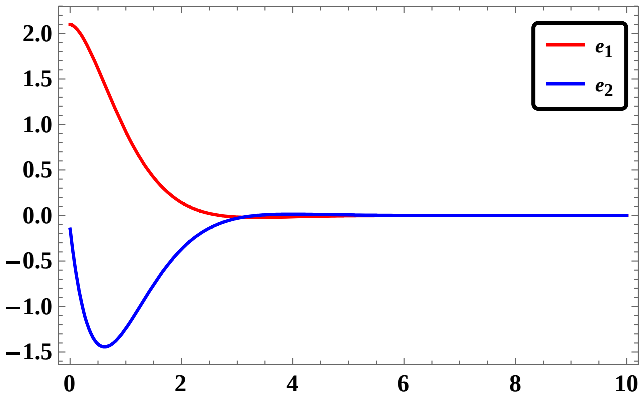

Figure 1. Time Series of e_1 & e_2.

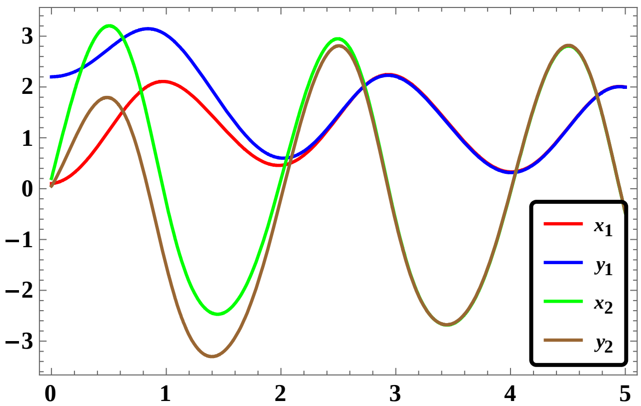

Figure 2. Time Series of x_1, y_1, x_2 & y_2.

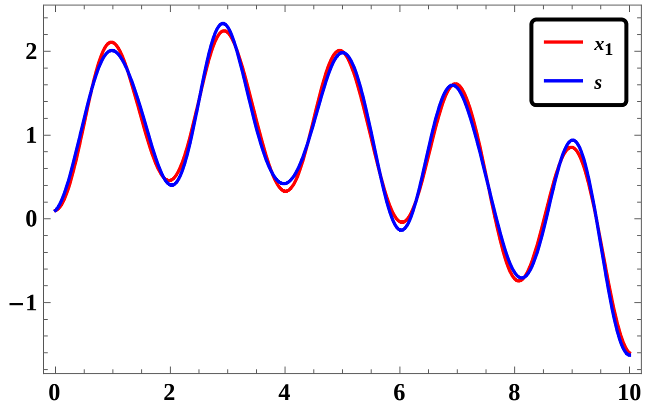

Figure 3. Time Series of x_1 & s.

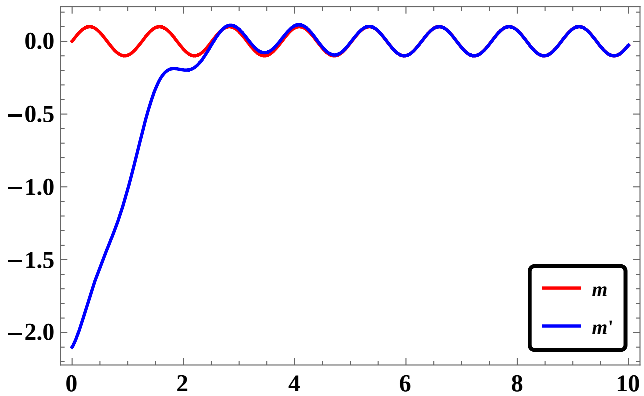

Figure 4. Time Series of m & m'.

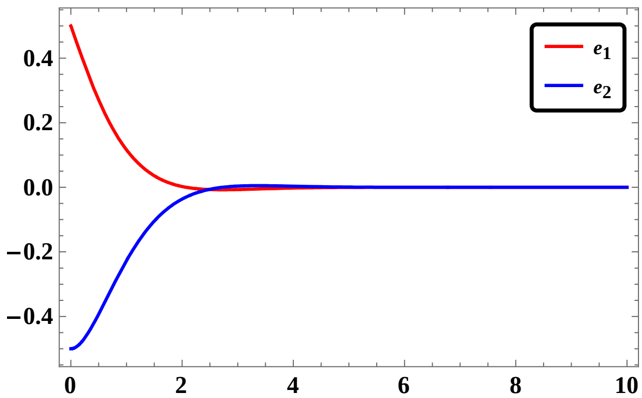

Figure 5. Time Series of e_1 & e_2.

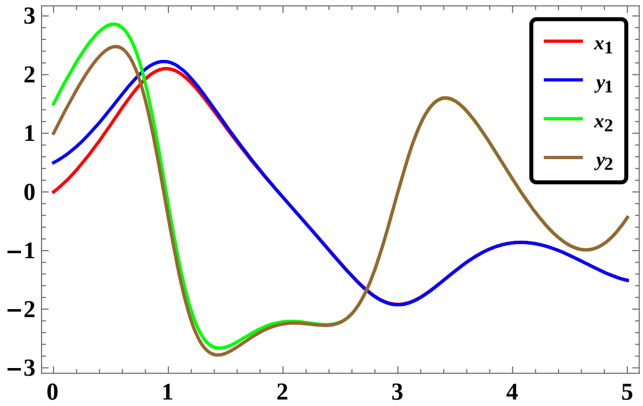

Figure 6. Time Series of x_1, y_1, x_2 & y_2.

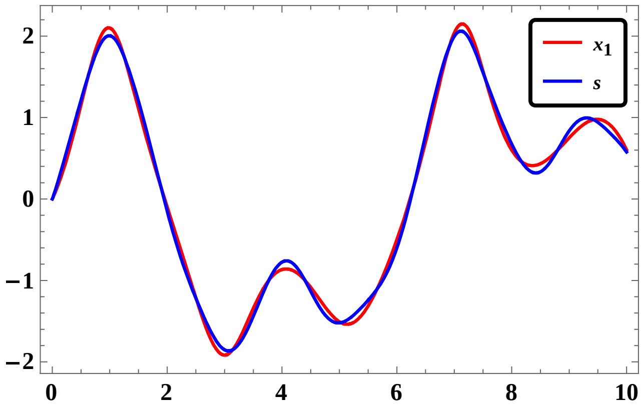

Figure 7. Time Series of x_1 & s.

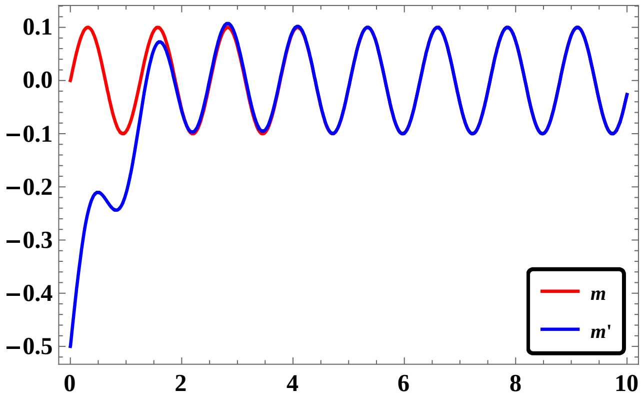

Figure 8. Time Series of m & m'.

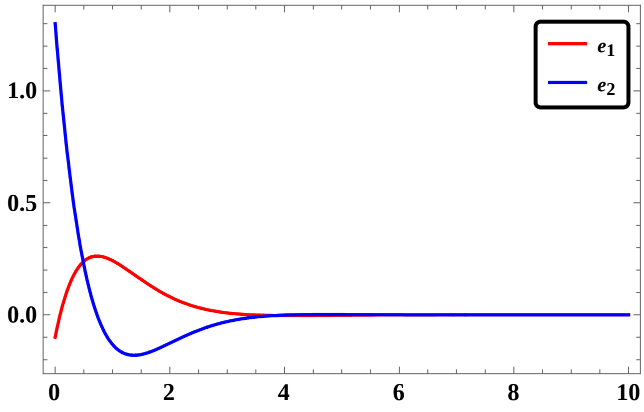

Figure 9. Time Series of e_1 & e_2.

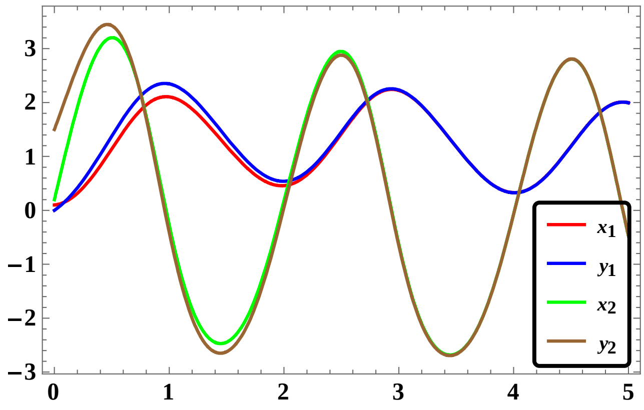

Figure 10. Time Series of x_1, y_1, x_2 & y_2.

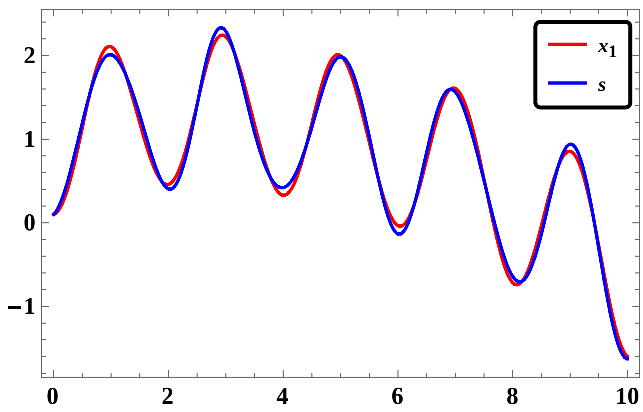

Figure 11. Time Series of x_1 & s.

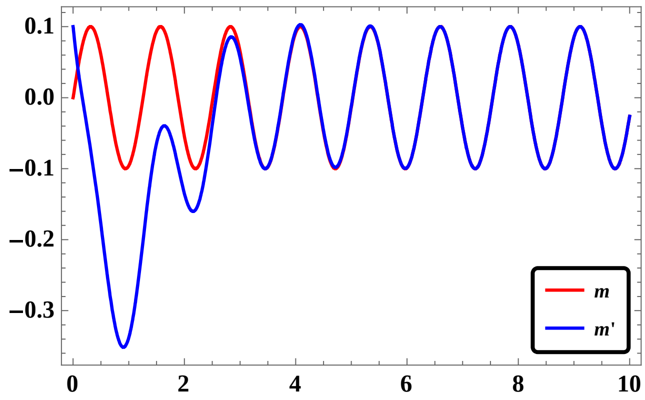

Figure 12. Time Series of m & m'.

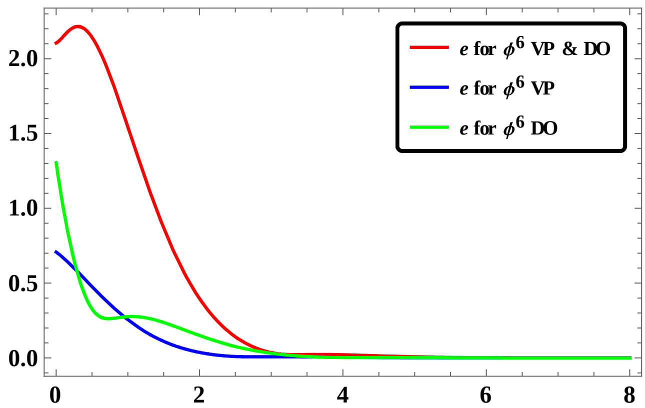

Figure 13. Time Series of e.

3. Synchronization Using RASMC

In this section, we synchronize the identical and non-identical pairs of Φ^6—VDPO and DO using RASMC technique that has a very quick response in stabilizing the synchronization of chaotic systems [32–36]. Below we describe the technique in details.

3.1. Description of RASMC

For the n-dimensional master and slave systems with external uncertainties, disturbances and unknown parameters, the RASMC [24] is described as follows:

$$\textbf{Master system}: \dot{x}(t) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x})\boldsymbol{\theta} + \Delta\boldsymbol{f}(\boldsymbol{x},t) + \boldsymbol{d}^m(t) \tag{13}$$

$$\textbf{Slave System}: \dot{\boldsymbol{y}}(t) = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{G}(\boldsymbol{y})\boldsymbol{\psi} + \Delta\boldsymbol{g}(\boldsymbol{y},t) + \boldsymbol{d}^s(t) + \boldsymbol{u}(t) \tag{14}$$

where $\boldsymbol{x}(t) = [x_1, x_2, \ldots, x_n]^T$ are the state vectors, $\boldsymbol{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})]^T$ are the continuous nonlinear functions, $F_i(\boldsymbol{x})$, $i = 1, 2, \ldots, n$, is i^{th} row of an $n \times n$ matrix $(\boldsymbol{F}(\boldsymbol{x}))$ whose elements are continuous nonlinear functions, $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_n]^T$ are the unknown vector parameters, and $\Delta\boldsymbol{f}(\boldsymbol{x},t) = [\Delta f_1(\boldsymbol{x},t), \Delta f_2(\boldsymbol{x},t), \ldots, \Delta f_n(\boldsymbol{x},t)]^T$ and $\boldsymbol{d}^m(t) = \left[d_1^m(t), d_2^m(t), \ldots, d_n^m(t)\right]^T$ are the vectors of unknown uncertainties and external disturbances of the master system, respectively. $\boldsymbol{y}(t) = [y_1, y_2, \ldots, y_n]^T$ are the state vectors, $\boldsymbol{g}(\boldsymbol{y}) = [g_1(\boldsymbol{y}), g_2(\boldsymbol{y}), \ldots, g_n(\boldsymbol{y})]^T$ are the continuous nonlinear functions, $G_i(\boldsymbol{y})$, $i = 1, 2, \ldots, n$, is i^{th} row of an $n \times n$ matrix $(\boldsymbol{G}(\boldsymbol{y}))$ whose elements are continuous nonlinear functions, $\boldsymbol{\psi} = [\psi_1, \psi_2, \ldots, \psi_n]^T$ are the unknown vector parameters, $\Delta\boldsymbol{g}(\boldsymbol{y},t) = [\Delta g_1(\boldsymbol{y},t), \Delta g_2(\boldsymbol{y},t), \ldots, \Delta g_n(\boldsymbol{y},t)]^T$ and $\boldsymbol{d}^s(t) = \left[d_1^s(t), d_2^s(t), \ldots, d_n^s(t)\right]^T$ are the vectors of unknown uncertainties and external disturbances of the slave system, respectively, and $\boldsymbol{u}(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T$ is the vector of control inputs.

Assumption 1: Since the trajectories of chaotic systems are always bounded, then the unknown uncertainties $\Delta \boldsymbol{f}(\boldsymbol{x}, t)$ and $\Delta \boldsymbol{g}(\boldsymbol{y}, t)$ are assumed to be bounded. Therefore, there exist appropriate positive constants α_i^m and α_i^s, $i = 1, 2, \ldots, n$ such that

$$|\Delta f_i(\boldsymbol{x}, t)| < \alpha_i^m \quad \text{and} \quad |\Delta g_i(\boldsymbol{y}, t)| < \alpha_i^s, \qquad i = 1, 2, \ldots n \tag{15}$$

$$\Rightarrow |\Delta f_i(\boldsymbol{x}, t) - \Delta g_i(\boldsymbol{y}, t)| < \alpha_i, \qquad i = 1, 2, \ldots, n, \text{ where } \alpha_i \text{ are unknown constants} \tag{16}$$

Assumption 2: In general, it is assumed that the external disturbances are norm-bounded in C^1,

$$i.e., \ |d_i^m(t)| < \beta_i^m \quad \text{and} \ |d_i^s(t)| < \beta_i^s, \qquad i = 1, 2, \ldots, n \tag{17}$$

$$\Rightarrow |d_i^m(t) - d_i^s(t)| < \beta_i, \qquad i = 1, 2, \ldots, n, \text{ where } \beta_i \text{ are unknown constants} \tag{18}$$

To solve the synchronization problem, the error between the master system (Equation (13)) and slave systems (Equation (14)) can be defined as $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t)$. Then, from Equations (13) and (14), the error dynamics can be written as:

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{f}(x) + \boldsymbol{F}(\boldsymbol{x})\boldsymbol{\theta} + \Delta \boldsymbol{f}(\boldsymbol{x}, t) + \boldsymbol{d}^m(t) - \boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{G}(\boldsymbol{y})\boldsymbol{\psi} - \Delta \boldsymbol{f}(\boldsymbol{y}, t) - \boldsymbol{d}^s(t) - \boldsymbol{u}(t) \tag{19}$$

It is clear that the synchronization problem can be transformed to the equivalent problem of stabilizing the error system (Equation (19)). The objective of this paper is to show that for any given master chaotic system (Equation (13)) and slave chaotic system (Equation (14)) with the uncertainties, external disturbances and unknown parameters a suitable feedback control law $\boldsymbol{u}(t)$ is designed such that the asymptotical stability of the resulting error system (Equation (19)) can be achieved in the sense that $\lim\limits_{t\to\infty} |\boldsymbol{x}(t) - \boldsymbol{y}(t)| \to 0$ is for the systems under consideration.

Let us consider now the appropriate sliding surface with the desired behavior. Therefore, the sliding surface suitable for the technique can be designed as:

$$s_i(t) = \lambda_i e_i(t), \quad i = 1, 2, \ldots, n \tag{20}$$

where $s_i(t) \in \mathbb{R}\,(\boldsymbol{s}(t) = [s_1(t), s_2(t), \ldots, s_n(t)])$ and the sliding surface parameters λ_i are positive constants.

After designing the suitable sliding surface, let us determine the input control signal $\boldsymbol{u}(t)$ to guarantee that the error system trajectories reach to the sliding surface $\boldsymbol{s}(t) = 0$ (*i.e.*, to satisfy the reaching condition $\boldsymbol{s}(t)\dot{\boldsymbol{s}}(t) < 0$) and stay on it permanently. Therefore, to ensure the existence of the sliding motion a discontinuous control law (with minimum chattering effect) is proposed as:

$$u_i(t) = f_i(\boldsymbol{x}) - g_i(\boldsymbol{y}) + F_i(\boldsymbol{x})\hat{\theta}_i - G_i(\boldsymbol{y})\hat{\psi}_i + \left(\hat{\alpha}_i + \hat{\beta}_i\right)\text{sgn}(s_i) + k_i \tanh(\varepsilon s_i), \quad \text{for } i = 1, 2, \ldots, n \tag{21}$$

where $\hat{\theta}_i$, $\hat{\psi}_i$, $\hat{\alpha}_i$, and $\hat{\beta}_i$ are estimations for θ_i, ψ_i, and α_i, respectively, $k_i > 0$, $\quad i = 1, 2, \ldots, n$ are the switching gain constant, and $\varepsilon > 0$.

To tackle the uncertainties, external disturbances and unknown parameters, appropriate update laws are defined as:

$$\begin{aligned}
&\dot{\hat{\boldsymbol{\theta}}} = [\boldsymbol{F}(\boldsymbol{x})]^T \gamma, && \hat{\boldsymbol{\theta}}(0) = \hat{\boldsymbol{\theta}}_0 \\
&\dot{\hat{\boldsymbol{\psi}}} = -[\boldsymbol{G}(\boldsymbol{y})]^T \gamma, && \hat{\boldsymbol{\psi}}(0) = \hat{\boldsymbol{\psi}}_0 \\
&\dot{\hat{\alpha}}_i = \dot{\hat{\beta}}_i = \lambda_i |s_i|, && \hat{\alpha}_i(0) = \hat{\alpha}_{i0} \ \& \ \hat{\beta}_i(0) = \hat{\beta}_{i0}
\end{aligned} \tag{22}$$

where $\gamma = [\lambda_1 s_1, \lambda_2 s_2, \ldots, \lambda_n s_n]^T$ and $\hat{\boldsymbol{\theta}}_0$, $\hat{\boldsymbol{\psi}}_0$, $\hat{\alpha}_{i0}$ and $\hat{\beta}_{i0}$ are the initial values of the update parameters $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\psi}}$, $\hat{\alpha}_i$ and $\hat{\beta}_i$, respectively.

Based on the control input in Equation (21) and update laws in Equation (22) as used to guarantee the reaching condition $\boldsymbol{s}(t)\dot{\boldsymbol{s}}(t) < 0$ and to ensure the occurrence of the sliding motion, we have the following theorem.

Theorem 1: Consider the error dynamics in Equation (19), this system is controlled by $\boldsymbol{u}(t)$ in Equation (13) with update laws in Equation (14). Then the error system trajectories will converge to the sliding surface $\boldsymbol{s}(t) = 0$.

In this regard, we consider a Lyapunov function (that is a positive definite function also) as follow:

$$V(t) = \frac{1}{2}\sum_{i=1}^{n}\left[s_i^2 + (\hat{\alpha}_i - \alpha_i)^2 + \left(\hat{\beta}_i - \beta_i\right)^2\right] + \frac{1}{2}||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}||^2 + \frac{1}{2}||\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}||^2 \tag{23}$$

In order to apply the RASMC to synchronize the identical pairs of chaotic Φ^6—VDPO; Φ^6—DO and non-identical pair Φ^6—VDPO and DO, external uncertainty for master and slave systems have been chosen as: $\Delta f_i(x_i, t) \to 0.5\sin x_i$ and $\Delta g_i(y_i, t) \to -0.5\sin y_i$, respectively; external disturbance for master and slave system: $d_i^m(t) \to 0.1\sin t$ and $d_i^s(t) \to -0.1\sin t$ for all $i = 1, 2$, respectively; $\varepsilon = 100$; initial values of update parameters $\hat{\theta}_i(0) = 1$, $\hat{\psi}_i(0) = 2$, $\hat{\alpha}_i(0) = 3$ and $\hat{\beta}_i(0) = 4$; and for secure communication message signal $(m) = 0.05\sin 2t$.

3.2. Synchronization of Two Identical Φ^6—VDPO Using RASMC

In this section, we synchronize the identical pairs of chaotic Φ^6—VDPO using RASMC under the effect of external uncertainty and external disturbance for both master and slave systems. After adding the external uncertainty and disturbances, Equation (1) can be written as a pair of master and slave systems:

$$\dot{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 \\ A \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix}}_{F(x,t)}\underbrace{\begin{bmatrix} 1 \\ \mu_1 \end{bmatrix}}_{\boldsymbol{\theta}} + \underbrace{\begin{bmatrix} 0.5\sin x_1 \\ 0.5\sin x_2 \end{bmatrix}}_{\Delta f(x,t)} + \underbrace{\begin{bmatrix} 0.1\sin t \\ 0.1\sin t \end{bmatrix}}_{d^m(t)} \tag{24}$$

$$\dot{\boldsymbol{y}} = \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{g(y)} + \underbrace{\begin{bmatrix} y_2 & 0 \\ 0 & y_2 \end{bmatrix}}_{G(y,t)}\underbrace{\begin{bmatrix} 1 \\ \mu_1 \end{bmatrix}}_{\boldsymbol{\psi}} + \underbrace{\begin{bmatrix} -0.5\sin y_1 \\ -0.5\sin y_2 \end{bmatrix}}_{\Delta g(y,t)} + \underbrace{\begin{bmatrix} -0.1\sin t \\ -0.1\sin t \end{bmatrix}}_{d^m(t)} + \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\boldsymbol{u}(t)} \tag{25}$$

where $A = -\mu_1 x_1^2 x_2 - \alpha_1(1 + \eta_1\cos 2\omega_1 t)x_1 - \beta_1 x_1^3 - \lambda_1 x_1^5 + f_1\cos\omega_1 t$, $B = -\mu_1 y_1^2 y_2 - \alpha_1(1 + \eta_1\cos 2\omega_1 t)y_1 - \beta_1 y_1^3 - \lambda_1 y_1^5 + f_1\cos\omega_1 t$ and $u_i(t)$ for $i = 1, 2$ are the controllers which govern as per the rule (Equation (21)). Furthermore, during simulation, the initial values of states vectors in master and slave systems are chosen as: $x_1(0) = 0.1$, $x_2(0) = 0.2$, $y_1(0) = 2.2$, $y_2(0) = 0.05$, respectively; sliding surface parameters: $\lambda_1 = 25$, $\lambda_2 = 10$ and switching gain constants: $k_1 = 1$, $k_2 = 5$.

Therefore, using Equation (19), the error dynamics can be expressed as:

$$\begin{aligned} \dot{e}_1 = &\ e_2 + 0.5(\sin x_1 + \sin y_1) + 0.2\sin t - u_1(t), \\ \dot{e}_2 = &\ \mu_1 e_1 - \mu_1(x_1^2 x_2 + y_1^2 y_2) - \alpha_1\{1 + \eta_1\cos(2\omega_1 t)\}e_1 - \beta_1(x_1^3 - y_1^3) \\ &-\lambda_1(x_1^5 - y_1^5) + 0.5(\sin x_2 + \sin y_2) + 0.2\sin t - u_2(t). \end{aligned} \tag{26}$$

where $u_i(t) = f_i(x) - g_i(y) + F_i(x)\hat{\theta}_i - G_i(y)\hat{\psi}_i + \left(\hat{\alpha}_i + \hat{\beta}_i\right)\mathrm{sign}(s_i) + k_i\tanh(\varepsilon s_i)$ for $i = 1, 2$ and the unknown parameters have been taken as per Equation (22). The following are the time series of synchronization errors (Figure 14), update parameters (Figures 15 and 16), states vectors (Figure 17) and for secure communication scheme (Figures 18 and 19).

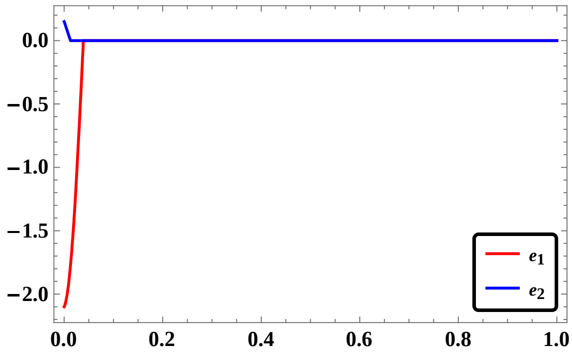

Figure 14. Time Series of e_1 & e_2.

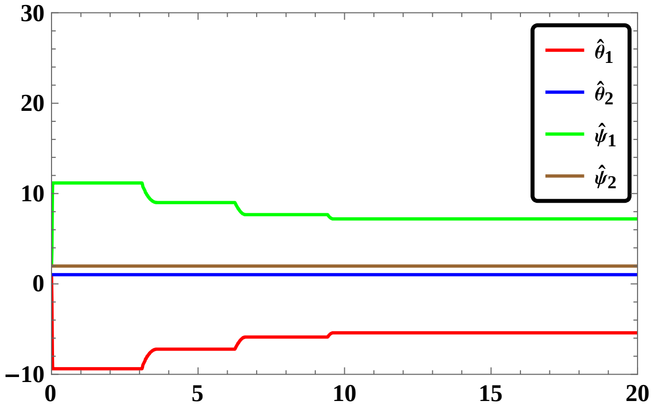

Figure 15. Time Series of $\hat{\theta}_1, \hat{\theta}_2, \hat{\psi}_1$ & $\hat{\psi}_2$.

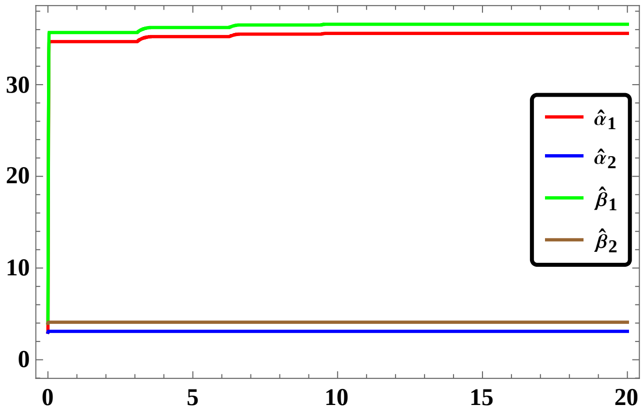

Figure 16. Time Series of $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1$ & $\hat{\beta}_2$.

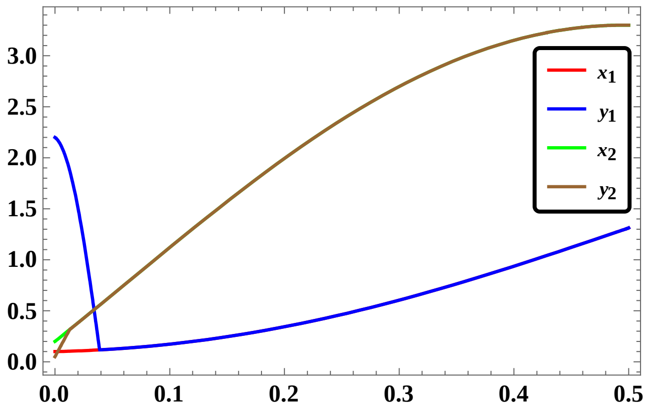

Figure 17. Time Series of x_1, y_1, x_2 & y_2.

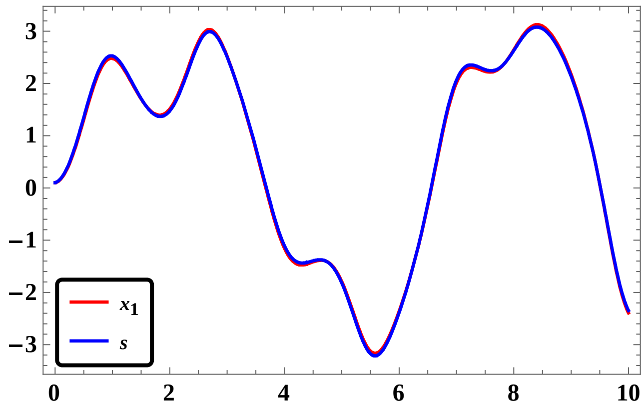

Figure 18. Time Series of x_1 & s.

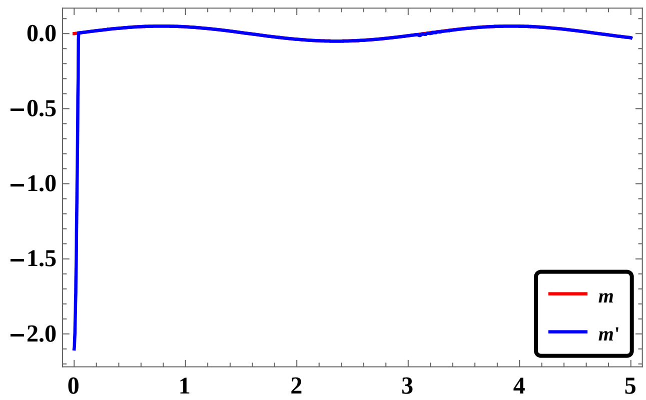

Figure 19. Time Series of m & m'.

3.3. *Synchronization of Two Identical* Φ^6*—DO Using RASMC*

In this section, we synchronize the identical pairs of chaotic Φ^6—DO using RASMC under the effect of external uncertainty and external disturbance for both master and slave systems. After adding the external uncertainty and disturbances, Equation (2) can be written as a pair of master and slave systems:

$$\dot{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 \\ A \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} x_2 & 0 \\ 0 & -x_2 \end{bmatrix}}_{F(x,t)} \underbrace{\begin{bmatrix} 1 \\ \mu_2 \end{bmatrix}}_{\theta} + \underbrace{\begin{bmatrix} 0.5\sin x_1 \\ 0.5\sin x_2 \end{bmatrix}}_{\Delta f(x,t)} + \underbrace{\begin{bmatrix} 0.1\sin t \\ 0.1\sin t \end{bmatrix}}_{d^m(t)} \tag{27}$$

$$\dot{y} = \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{g(y)} + \underbrace{\begin{bmatrix} y_2 & 0 \\ 0 & -y_2 \end{bmatrix}}_{G(y,t)} \underbrace{\begin{bmatrix} 1 \\ \mu_2 \end{bmatrix}}_{\psi} + \underbrace{\begin{bmatrix} -0.5\sin y_1 \\ -0.5\sin y_2 \end{bmatrix}}_{\Delta g(y,t)} + \underbrace{\begin{bmatrix} -0.1\sin t \\ -0.1\sin t \end{bmatrix}}_{d^m(t)} + \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{u(t)} \tag{28}$$

where

$$A = -\alpha_2\left\{1 + \eta_2\cos(2\omega_2 t)\right\} x_1 - \beta_2 x_1^3 - \lambda_2 x_1^5 + f_2\cos\omega_2 t,$$

$$B = -\alpha_2\left\{1 + \eta_2\cos(2\omega_2 t)\right\} y_1 - \beta_2 y_1^3 - \lambda_2 y_1^5 + f_2\cos\omega_2 t$$

and $u_i(t)$ for $i = 1, 2$ are the controllers which govern as per the rule (Equation (22)). Furthermore, during simulation, the initial values of states vectors in master and slave systems are chosen as: $x_1(0) = 0$, $x_2(0) = 1.5$, $y_1(0) = 0.5$, $y_2(0) = 1$, respectively; sliding surface parameters: $\lambda_1 = 12$, $\lambda_2 = 10$; and switching gain constants: $k_1 = k_2 = 20$.

Therefore, using Equation (19), the error dynamics can be expressed as:

$$\begin{aligned} \dot{e}_1 = &\; e_2 + 0.5\left(\sin x_1 + \sin y_1\right) + 0.2\sin t - u_1(t), \\ \dot{e}_2 = &\; -\mu_2 e_1 - \alpha_2\left\{1 + \eta_1\cos(2\omega_1 t)\right\} e_1 - \beta_1(x_1^3 - y_1^3) - \lambda_1(x_1^5 - y_1^5) \\ &+ 0.5\left(\sin x_2 + \sin y_2\right) + 0.2\sin t - u_2(t). \end{aligned} \tag{29}$$

where $u_i(t) = f_i(x) - g_i(y) + F_i(x)\hat{\theta}_i - G_i(y)\hat{\psi}_i + \left(\hat{\alpha}_i + \hat{\beta}_i\right)\text{sign}(s_i) + k_i\tanh(\varepsilon s_i)$ for $i = 1, 2$ and the unknown parameters have been taken as per (22). The following are the time series of synchronization errors (Figure 20), update parameters (Figures 21 and 22), states vectors (Figure 23) and for secure communication scheme (Figures 24 and 25).

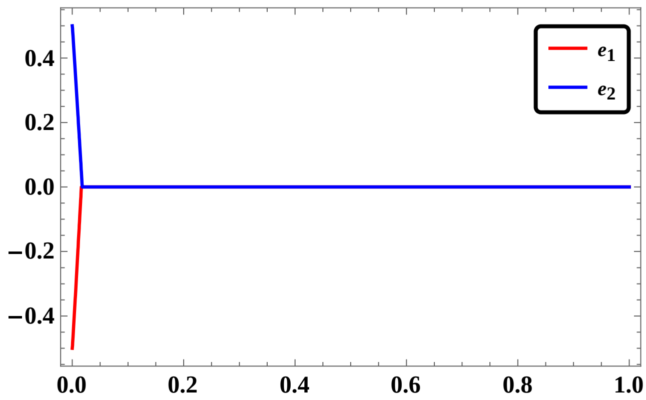

Figure 20. Time Series of e_1 & e_2.

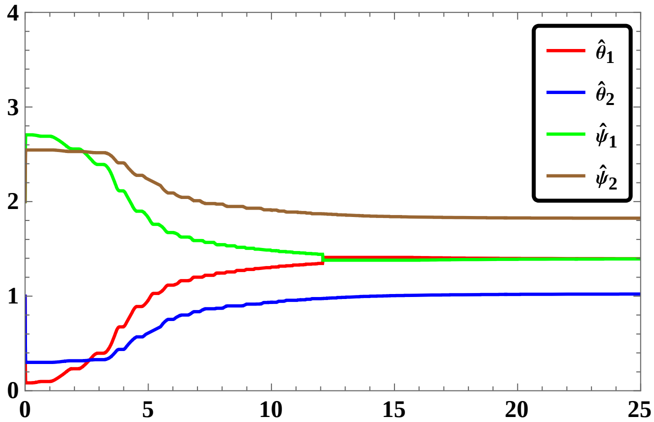

Figure 21. Time Series of $\hat{\theta}_1, \hat{\theta}_2, \hat{\psi}_1$ & $\hat{\psi}_2$.

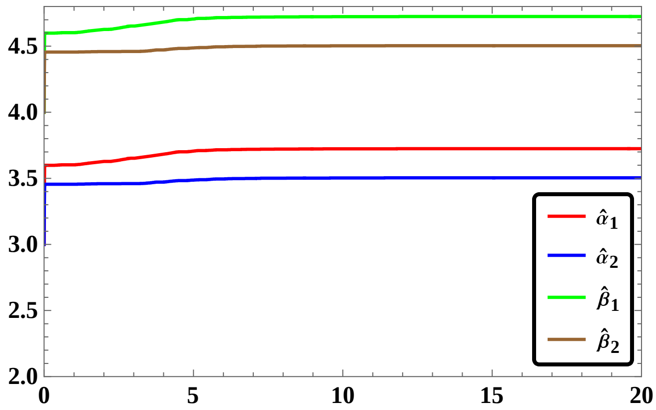

Figure 22. Time Series of $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1$ & $\hat{\beta}_2$.

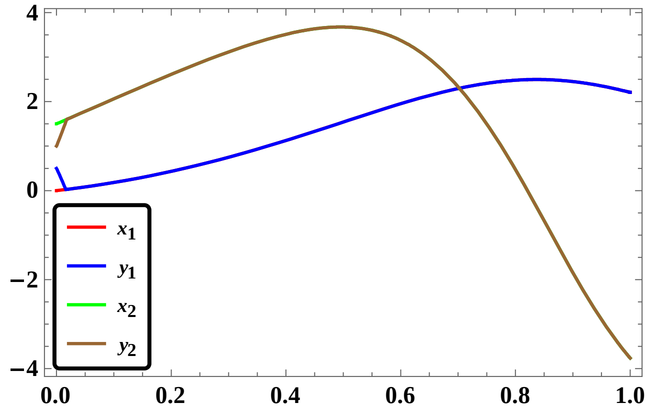

Figure 23. Time Series of x_1, y_1, x_2 & y_2.

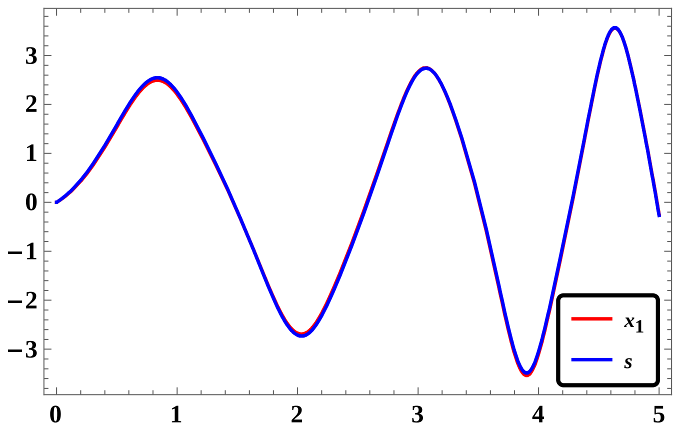

Figure 24. Time Series of x_1 & s.

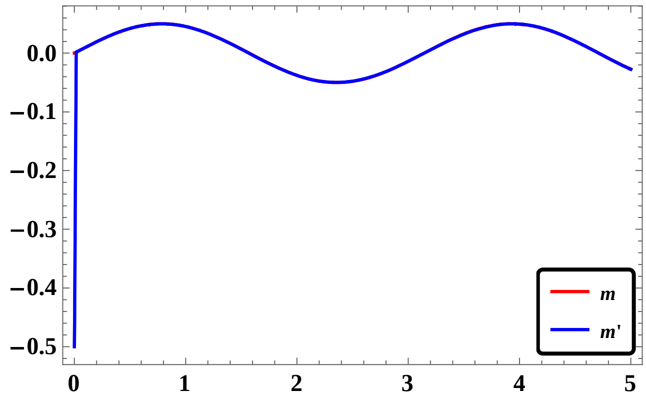

Figure 25. Time Series of m & m'.

3.4. Synchronization of Φ^6—VDPO and DO Using RASMC

In this section, we synchronize the non-identical pairs of chaotic Φ^6—VDPO and DO using RASMC under the effect of external uncertainty and external disturbance for both master and slave systems. After adding the external uncertainty and disturbances, Equations (1) and (2) can be written as a pair of master and slave systems, respectively:

$$\dot{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 \\ A \end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{x})} + \underbrace{\begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix}}_{\boldsymbol{F}(\boldsymbol{x},t)} \underbrace{\begin{bmatrix} 1 \\ \mu_1 \end{bmatrix}}_{\boldsymbol{\theta}} + \underbrace{\begin{bmatrix} 0.5\sin x_1 \\ 0.5\sin x_2 \end{bmatrix}}_{\Delta \boldsymbol{f}(\boldsymbol{x},t)} + \underbrace{\begin{bmatrix} 0.1\sin t \\ 0.1\sin t \end{bmatrix}}_{\boldsymbol{d}^m(t)} \tag{30}$$

$$\dot{\boldsymbol{y}} = \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\boldsymbol{g}(\boldsymbol{y})} + \underbrace{\begin{bmatrix} y_2 & 0 \\ 0 & -y_2 \end{bmatrix}}_{\boldsymbol{G}(\boldsymbol{y},t)} \underbrace{\begin{bmatrix} 1 \\ \mu_2 \end{bmatrix}}_{\boldsymbol{\psi}} + \underbrace{\begin{bmatrix} -0.5\sin y_1 \\ -0.5\sin y_2 \end{bmatrix}}_{\Delta \boldsymbol{g}(\boldsymbol{y},t)} + \underbrace{\begin{bmatrix} -0.1\sin t \\ -0.1\sin t \end{bmatrix}}_{\boldsymbol{d}^m(t)} + \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\boldsymbol{u}(t)} \tag{31}$$

where

$$A = -\mu_1 x_1^2 x_2 - \alpha_1 \{1 + \eta_1 \cos(2\omega_1 t)\} x_1 - \beta_1 x_1^3 - \lambda_1 x_1^5 + f_1 \cos\omega_1 t$$

$$B = -\alpha_2 \{1 + \eta_2 \cos(2\omega_2 t)\} y_1 - \beta_2 y_1^3 - \lambda_2 y_1^5 + f_2 \cos\omega_2 t$$

and $u_i(t)$ for $i = 1,2$ are the controllers which govern as per the rule (Equation (22)). Furthermore, during simulation, the initial values of states vectors in master and slave systems are chosen as: $x_1(0) = 0.1$, $x_2(0) = 0.2$, $y_1(0) = 0$, and $y_2(0) = 1.5$, respectively; sliding surface parameters: $\lambda_1 = 12$, $\lambda_2 = 10$; and switching gain constants: $k_1 = k_2 = 20$.

Therefore, using Equation (19), the error dynamics can be expressed as:

$$\begin{aligned} \dot{e}_1 = {} & e_2 + 0.5(\sin x_1 + \sin y_1) + 0.2\sin t - u_1(t), \\ \dot{e}_2 = {} & \mu_1(1 - x_1^2)x_2 - \alpha_1 \{1 + \eta_1 \cos(2\omega_1 t)\} x_1 - \beta_1 x_1^3 + \beta_2 y_1^3 - \lambda_1 x_1^5 + \lambda_2 y_1^5 + f_1 \cos\omega_1 t \\ & + \mu_2 y_2 + \alpha_2 \{1 + \eta_1 \cos(2\omega_2 t)\} y_1 + f_2 \cos\omega_2 t + 0.5(\sin x_2 + \sin y_2) + 0.2\sin t - u_2(t). \end{aligned} \tag{32}$$

where $u_i(t) = f_i(x) - g_i(y) + F_i(x)\hat{\theta}_i - G_i(y)\hat{\psi}_i + (\hat{\alpha}_i + \hat{\beta}_i)\,\mathrm{sign}(s_i) + k_i \tanh(\varepsilon s_i)$ for $i = 1,2$ and the unknown parameters have been taken as per Equation (22). The following are the time series of synchronization errors (Figure 26), update parameters (Figures 27 and 28), states vectors (Figure 29) and for secure communication scheme (Figures 30 and 31) as well as the derivative of Lyapunov function ($\dot{V}(t)$) for all three pairs together (Figure 32).

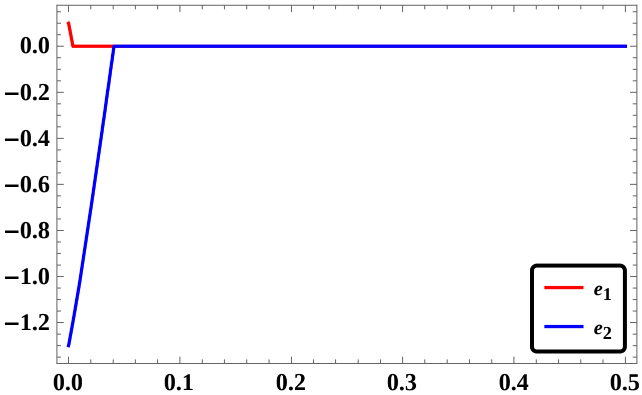

Figure 26. Time Series of e_1 & e_2.

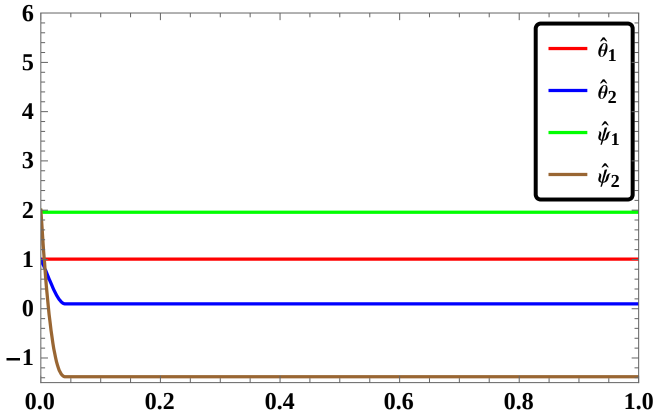

Figure 27. Time Series of $\hat{\theta}_1, \hat{\theta}_2, \hat{\psi}_1$ & $\hat{\psi}_2$.

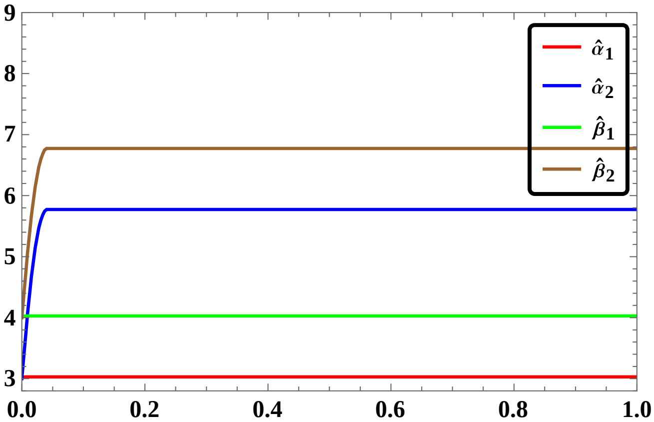

Figure 28. Time Series of $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1$ & $\hat{\beta}_2$.

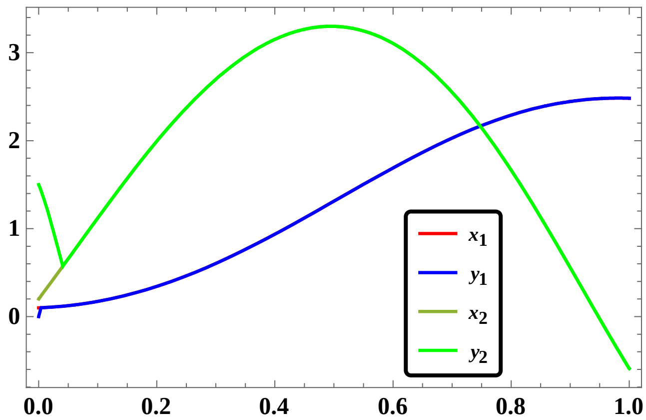

Figure 29. Time Series of x_1, y_1, x_2 & y_2.

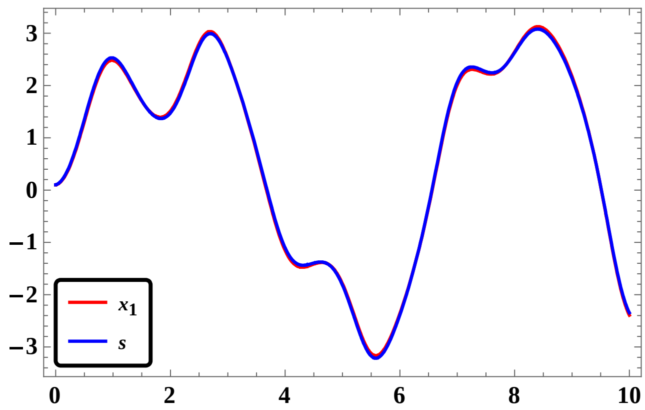

Figure 30. Time Series of x_1 & s.

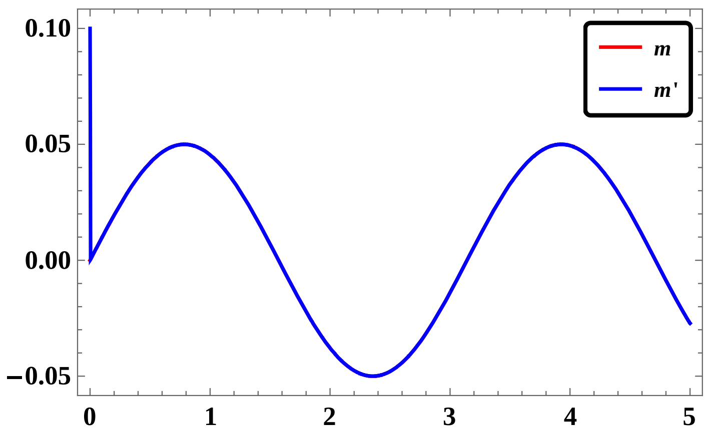

Figure 31. Time Series of m & m'.

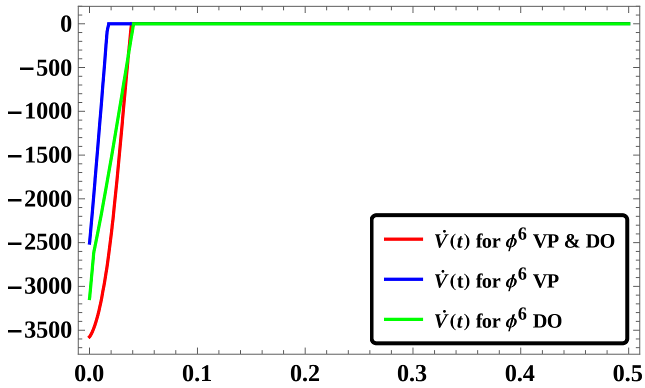

Figure 32. Time Series of $\dot{V}(t)$ in all cases.

3.5. Numerical Simulations and Discussion

The main aim of Section 3 was to implement the RASMC on the two identical pairs of Φ^6—VDPO and DO and one non-identical pair of Φ^6—VDPO and DO for synchronization purpose. It has been observed that not only RASMC is found to be very effective for all the three pairs under consideration; it is also effective for a secure communication scheme. To the best of our knowledge, this has been done for the first time using RASMC. The time of stabilization of synchronization is very short (nearly $t = 0.2$) for all the cases that can be observed in the plotted time series (Figures 14–32). We can say now that the time response of stabilization of synchronization is much quicker in RASMC technique than it was in LAC.

4. Conclusions

In this computational cum comparative study, the problem of chaotic synchronization of chaotic systems is repeated using *Mathematica* via LAC technique. It has been found that the time response of stabilization of synchronization is reduced by half when it is done using *Mathematica*. On the other hand, when the same pairs are synchronized via RAMSC, the time response of stabilization of synchronization is found to be much faster than the LAC technique. Finally, we conclude the following remarkable features of our proposed study:

(1) The time response of stabilization of synchronization for LAC in our study was found to occur with rapid convergence if simulation is done with *Mathematica*.
(2) For the same pairs of master and slave systems considered in our study, the RASMC is found to be more effective in terms of time response of stabilization of synchronization.

On the basis of these two points, we conclude that selection of appropriate mathematical tools for simulation and technique for synchronization is very important.

Acknowledgments: The authors would like to express their gratitude to the honorable reviewers who suggested many worthwhile changes to improve the work of this manuscript.

Author Contributions: Azizan Bin Saaban performed the literature review and proposed the problem. Adyda Binti Ibrahin and Israr Ahmad performed the analytical analysis and wrote the paper. Mohammad Shahzad performed the numerical simulations and revised the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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