OPERATION-SEPARATION AXIOMS VIA $\gamma$-$P_S$-OPEN SETS

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Abstract: This paper defines some new $\gamma$-$P_S$- separation axioms called $\gamma$-$P_S$-$T_i$ for $i = 0, 1, 2$ using $\gamma$-$P_S$-open sets. Some theoretical results and properties for these $\gamma$-$P_S$- separation axioms are obtained. Several examples are given to illustrate some of the results.

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1. Introduction

The concept of $\alpha$-closed graphs of an operation on $\tau$ had been defined by Kasahara [8]. The operation $\alpha$ had been renamed as $\gamma$ operation on $\tau$ by Ogata [11]. He defined $\gamma$-open sets and introduced the notion of $\tau_\gamma$ which is the class of all $\gamma$-open sets in a topological space $(X, \tau)$. Since that, $\gamma$ operation on $\tau$ has attracted the attention of many researchers. Among them are $\gamma$-preopen sets [9], $\gamma$-semiopen sets [10] and $\gamma$-$\beta$-open sets [6]. Using these sets, several separation...
axioms such as $\gamma$-pre$T_i$ [9], $\gamma$-semi$T_i$ [10] and $\gamma$-$\beta T_i$ [6] for $i = 0, \frac{1}{2}, 1, 2$ were defined.

Recently, the notion of $\gamma$-$P_S$-open sets had been defined by Asaad, Ahmad and Omar [2]. This set is stronger than $\gamma$-preopen set. Besides that, they also introduced $\gamma$-locally indiscrete and $\gamma$-hyperconnected spaces [1]. In addition, they introduced and studied the notion of $\gamma$-$P_S$-$T_{\frac{3}{2}}$ space [4].

The aim of this paper is to introduce some new $\gamma$-$P_S$- separation axioms called $\gamma$-$P_S$-$T_i$ for $i = 0, 1, 2$ by using $\gamma$-$P_S$-open sets. Some relations between these constructed $\gamma$-$P_S$- spaces with other existence $\gamma$- separation axioms as well as their properties have also been investigated.

2. Preliminaries and Main Definitions

In this paper, the pairs $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) always mean topological spaces on which no separation axioms assumed unless otherwise mentioned. An operation $\gamma$ on the topology $\tau$ on $X$ is a mapping $\gamma: \tau \rightarrow P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of $X$ and $\gamma(U)$ denotes the value of $\gamma$ at $U$ [11]. A nonempty subset $A$ of a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-open [11] if for each $x \in A$, there exists an open set $U$ containing $x$ such that $\gamma(U) \subseteq A$. The complement of a $\gamma$-open set is called a $\gamma$-closed. The $\tau_\gamma$-closure of a subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is defined as the intersection of all $\gamma$-closed sets containing $A$ and it is denoted by $\tau_\gamma Cl(A)$ [11], and the $\tau_\gamma$-interior of a subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is defined as the union of all $\gamma$-open sets containing $A$ [10] and it is denoted by $\tau_\gamma \text{Int}(A)$ [10]. An operation $\gamma$ on $\tau$ is said to be regular if for every open neighborhood $U$ and $V$ of each $x \in X$, there exists an open neighborhood $W$ of $x$ such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$ [11].

Let’s recall some known notions which are useful in the sequel.

**Definition 2.1.** Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. A subset $A$ of $X$ is said to be $\gamma$-preopen [9] (respectively, $\gamma$-semiopen [10] and $\gamma$-$\beta$-open [6]) if $A \subseteq \tau_\gamma \text{Int}(\tau_\gamma Cl(A))$ (respectively, $A \subseteq \tau_\gamma Cl(\tau_\gamma \text{Int}(A))$ and $A \subseteq \tau_\gamma Cl(\tau_\gamma \text{Int}(\tau_\gamma Cl(A)))$).

**Definition 2.2.** The complement of a $\gamma$-preopen [9] (respectively, $\gamma$-semiopen [10] and $\gamma$-$\beta$-open [6]) set is called $\gamma$-preclosed (respectively, $\gamma$-semiclosed and $\gamma$-$\beta$-closed).

**Definition 2.3.** [2] A $\gamma$-preopen subset $A$ of a topological space $(X, \tau)$ is called $\gamma$-$P_S$-open if for each $x \in A$, there exists a $\gamma$-semiclosed set $F$ such that
Let $A$ be any subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $\tau$. Then the $\tau_\gamma$-closure [2] (respectively, $\tau_\gamma$-preclosure [9], $\tau_\gamma$-semiclosure [10] and $\tau_\gamma$-$\beta$-closure [6]) of $A$ is defined as the intersection of all $\gamma$-$P_S$-closed (respectively, $\gamma$-preclosed, $\gamma$-semiclosed and $\gamma$-$\beta$-closed) sets of $X$ containing $A$ and it is denoted by $\tau_\gamma$-$P_S\text{Cl}(A)$ (respectively, $\tau_\gamma$-$p\text{Cl}(A)$, $\tau_\gamma$-$s\text{Cl}(A)$ and $\tau_\gamma$-$\beta\text{Cl}(A)$).

**Lemma 2.5.** [2] Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $\tau$. Then:

1. $A$ is $\gamma$-$P_S$-closed if and only if $\tau_\gamma$-$P_S\text{Cl}(A) = A$.
2. $x \in \tau_\gamma$-$P_S\text{Cl}(A)$ if and only if $A \cap U \neq \emptyset$ for every $\gamma$-$P_S$-open set $U$ of $X$ containing $x$.

**Definition 2.6.** A subset $A$ of a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be:

1. $\gamma$-$P_S$-generalized closed ($\gamma$-$P_S$-$g$-closed) if $\tau_\gamma$-$P_S\text{Cl}(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is $\gamma$-$P_S$-open set in $X$ [4].
2. $\gamma$-pre-generalized closed ($\gamma$-$\text{preg}$-closed) if $\tau_\gamma$-$p\text{Cl}(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is $\gamma$-preopen set in $X$ [9].
3. $\gamma$-semi-generalized closed ($\gamma$-$\text{semig}$-closed) if $\tau_\gamma$-$s\text{Cl}(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is $\gamma$-semiopen set in $X$ [10].
4. $\gamma$-$\beta$-generalized closed ($\gamma$-$\beta$-$g$-closed) if $\tau_\gamma$-$\beta\text{Cl}(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is $\gamma$-$\beta$-open set in $X$ [6].

The class of all $\gamma$-$P_S$-open (respectively, $\gamma$-$P_S$-closed and $\gamma$-$P_S$-$g$-closed) sets of a topological space $(X, \tau)$ is denoted by $\tau_\gamma$-$P_S\text{O}(X)$ (respectively, $\tau_\gamma$-$P_S\text{C}(X)$ and $\tau_\gamma$-$P_S\text{GC}(X)$).

**Definition 2.7.** A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-$P_S$-$T_{\frac{1}{2}}$ [4] (respectively, $\gamma$-$\text{pre}T_{\frac{1}{2}}$ [9], $\gamma$-$\text{semi}T_{\frac{1}{2}}$ [10] and $\gamma$-$\beta T_{\frac{1}{2}}$ [6]) if every $\gamma$-$P_S$-$g$-closed (respectively, $\gamma$-$\text{preg}$-closed, $\gamma$-$\text{semig}$-closed and $\gamma$-$\beta g$-closed) set in $X$ is $\gamma$-$P_S$-closed (respectively, $\gamma$-preclosed, $\gamma$-semiopen and $\gamma$-$\beta$-open).

**Definition 2.8.** A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-$\text{pre}T_0$ [9] (respectively, $\gamma$-$\text{semi}T_0$ [10] and $\gamma$-$\beta T_0$ [6]) if for each pair of distinct points $x, y$ in $X$, there exists a $\gamma$-preopen (respectively, $\gamma$-semiopen and $\gamma$-$\beta$-open) set $U$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. 

$x \in F \subseteq A$. The complement of a $\gamma$-$P_S$-open set is called $\gamma$-$P_S$-closed.
**Definition 2.9.** A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is said to be \(\gamma\)-pre\(T_1\) [9] (respectively, \(\gamma\)-semi\(T_1\) [10] and \(\gamma\)-\(\beta\)\(T_1\) [6]) if for each pair of distinct points \(x, y\) in \(X\), there exist two \(\gamma\)-preopen (respectively, \(\gamma\)-semiopen and \(\gamma\)-\(\beta\)-open) sets \(U\) and \(V\) such that \(x \in U\) but \(y \notin U\) and \(y \in V\) but \(x \notin V\).

**Definition 2.10.** A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is said to be \(\gamma\)-pre\(T_2\) [9] (respectively, \(\gamma\)-semi\(T_2\) [10] and \(\gamma\)-\(\beta\)\(T_2\) [6]) if for each pair of distinct points \(x, y\) in \(X\), there exist two disjoint \(\gamma\)-preopen (respectively, \(\gamma\)-semiopen and \(\gamma\)-\(\beta\)-open) sets.

**Theorem 2.11.** [4] For any topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\). Then \(X\) is \(\gamma\)-\(PS\)-\(T_1\) if and only if for each element \(x \in X\), the set \(\{x\}\) is \(\gamma\)-\(PS\)-closed or \(\gamma\)-\(PS\)-open.

### 3. \(\gamma\)-\(PS\)-\(T_i\) Spaces for \(i = 0, 1, 2\)

In this section, we introduce some types of \(\gamma\)-\(PS\)-separation axioms called \(\gamma\)-\(PS\)-\(T_i\) for \(i = 0, 1, 2\) using \(\gamma\)-\(PS\)-open set. The relation between these \(\gamma\)-\(PS\)-spaces and other types of \(\gamma\)-spaces will be investigated. Some basic properties of them are studied.

**Definition 3.1.** A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is said to be:

1. \(\gamma\)-\(PS\)-\(T_0\) if for each pair of distinct points \(x, y\) in \(X\), there exists a \(\gamma\)-\(PS\)-open set \(G\) such that either \(x \in G\) and \(y \notin G\) or \(y \in G\) and \(x \notin G\).

2. \(\gamma\)-\(PS\)-\(T_1\) if for each pair of distinct points \(x, y\) in \(X\), there exist two \(\gamma\)-\(PS\)-open sets \(G\) and \(H\) such that \(x \in G\) but \(y \notin G\) and \(y \in H\) but \(x \notin H\).

3. \(\gamma\)-\(PS\)-\(T_2\) if for each pair of distinct points \(x, y\) in \(X\), there exist two \(\gamma\)-\(PS\)-open sets \(G\) and \(H\) containing \(x\) and \(y\) respectively such that \(G \cap H = \emptyset\).

The following are some basic properties of \(\gamma\)-\(PS\)-\(T_i\) spaces for \(i = 0, 1, 2\).

**Theorem 3.2.** Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). Then \(X\) is \(\gamma\)-\(PS\)-\(T_0\) if and only if \(\tau_{\gamma\text{-PS}\text{Cl}}(\{x\}) \neq \tau_{\gamma\text{-PS}\text{Cl}}(\{y\})\), for every pair of distinct points \(x, y\) of \(X\).

**Proof.** Let \((X, \tau)\) be a \(\gamma\)-\(PS\)-\(T_0\) and \(x, y\) be any two distinct points of \(X\). Then there exists a \(\gamma\)-\(PS\)-open set \(G\) containing \(x\) or \(y\) (say \(x\), but not \(y\)). Then
Theorem 3.3. Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). Then \(X\) is \(\gamma\)-\(P_S\)-\(T_1\) if and only if every singleton set in \(X\) is \(\gamma\)-\(P_S\)-closed.

Proof. Suppose \((X, \tau)\) be \(\gamma\)-\(P_S\)-\(T_1\). Let \(x \in X\). Then for any point \(y \in X\) such that \(x \neq y\), there exists a \(\gamma\)-\(P_S\)-open set \(G\) such that \(y \in G\) but \(x \notin G\). Thus, \(y \in G \subseteq X \setminus \{x\}\). This implies that \(X \setminus \{x\} = \bigcup \{G : y \in X \setminus \{x\}\}\). Since the union of \(\gamma\)-\(P_S\)-open sets is \(\gamma\)-\(P_S\)-open. Then \(X \setminus \{x\}\) is \(\gamma\)-\(P_S\)-open set in \(X\). Hence \(\{x\}\) is \(\gamma\)-\(P_S\)-closed set in \(X\).

Conversely, suppose every singleton set in \(X\) is \(\gamma\)-\(P_S\)-closed. Let \(x, y \in X\) such that \(x \neq y\). This implies that \(x \in X \setminus \{y\}\). By hypothesis, we get \(X \setminus \{y\}\) is a \(\gamma\)-\(P_S\)-open set contains \(x\) but not \(y\). Similarly \(X \setminus \{x\}\) is a \(\gamma\)-\(P_S\)-open set contains \(y\) but not \(x\). Therefore, a space \(X\) is \(\gamma\)-\(P_S\)-\(T_1\). □

Theorem 3.4. For a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\). The following conditions are equivalent.

1. \(X\) is \(\gamma\)-\(P_S\)-\(T_2\).

2. If \(x \in X\), then there exists a \(\gamma\)-\(P_S\)-open set \(G\) containing \(x\) such that \(y \notin \gamma-P_S\text{Cl}(G)\) for each \(y \in X\).

3. For each \(x \in X\), \(\bigcap \{\gamma-P_S\text{Cl}(G) : G\text{ a }\gamma\)-\(P_S\)-open set containing \(x\}\) = \(\{x\}\).

Proof. (1) \(\Rightarrow\) (2) Let \(X\) be any \(\gamma\)-\(P_S\)-\(T_2\) space. For each \(x, y \in X\) with \(x \neq y\), then there exist two \(\gamma\)-\(P_S\)-open sets \(G\) and \(H\) containing \(x\) and \(y\) respectively such that \(G \cap H = \emptyset\). This implies that \(G \subseteq X \setminus H\) and hence \(\gamma-P_S\text{Cl}(\{G\}) \subseteq X \setminus H\) since \(X \setminus H\) is \(\gamma\)-\(P_S\)-closed set in \(X\). Therefore, \(y \notin \gamma-P_S\text{Cl}(G)\).

(2) \(\Rightarrow\) (3) Straightforward.

(3) \(\Rightarrow\) (1) Let \(x, y \in X\) with \(x \neq y\), then by hypothesis there exists a \(\gamma\)-\(P_S\)-open set \(G\) containing \(x\) such that \(y \notin G\) and hence \(y \notin \gamma-P_S\text{Cl}(G)\).
Then \( y \in X \setminus \tau_\gamma P_S Cl(G) \) and \( X \setminus \tau_\gamma P_S Cl(G) \) is \( \gamma P_S \)-open set. So \( G \cap X \setminus \tau_\gamma P_S Cl(G) = \emptyset \). Therefore, \( X \) is \( \gamma P_S T_2 \) space.

The following remark follows directly from Definition 3.1 (3).

**Remark 3.5.** If for each pair of distinct points \( x, y \) in a topological space \((X, \tau)\), there exist two \( \gamma P_S \)-open sets \( G \) and \( H \) containing \( x \) and \( y \) respectively such that \( \tau_\gamma P_S Cl(G) \cap \tau_\gamma P_S Cl(H) = \emptyset \). Then \( X \) is \( \gamma P_S T_2 \).

The relations between the \( \gamma P_S T_i \) for \( i = 0, \frac{1}{2}, 1, 2 \) are given as follows:

**Lemma 3.6.** Let \((X, \tau)\) be a topological space and \( \gamma \) be an operation on \( \tau \). Then the following statements hold:

1. If \( X \) is \( \gamma P_S T_2 \), then it is \( \gamma P_S T_1 \).
2. If \( X \) is \( \gamma P_S T_1 \), then it is \( \gamma P_S T_{1 \frac{1}{2}} \).
3. If \( X \) is \( \gamma P_S T_{1 \frac{1}{2}} \), then it is \( \gamma P_S T_0 \).

**Proof.** (1) Directly from Definition 3.1 (2) and (3).
(2) Directly from Theorem 3.3 and Theorem 2.11.
(3) Let \( x, y \in X \) such that \( x \neq y \). Since \( X \) is \( \gamma P_S T_{1 \frac{1}{2}} \) space. Then by Theorem 2.11, the set \( \{x\} \) is either \( \gamma P_S \)-closed or \( \gamma P_S \)-open. If \( \{x\} \) is \( \gamma P_S \)-closed, then \( X \setminus \{x\} \) is \( \gamma P_S \)-open. Hence \( y \in X \setminus \{x\} \) and \( x \notin X \setminus \{x\} \). So \( X \) is \( \gamma P_S T_0 \). Or, if \( \{x\} \) is \( \gamma P_S \)-open. Then \( x \in \{x\} \) and \( y \notin \{x\} \) and hence \( X \) is \( \gamma P_S T_0 \) space.

The converse of the above remark does not always true as shown from the following example.

**Example 3.7.** Let \((X, \tau)\) be any infinite set with the cofinite topology and \( \gamma \) be an operation on \( \tau \). Define an operation \( \gamma \) on \( \tau \) by \( \gamma(A) = A \) for all \( A \in \tau \). Simplify the space \( X \) is \( \gamma P_S T_1 \) and hence it is \( \gamma \)-pre\( T_1 \) and \( \gamma \)-semi\( T_1 \). Then by Corollary 3.15, \( X \) is \( \gamma P_S T_1 \), but not \( \gamma P_S T_2 \), since for \( x \) and \( y \) in \( X \), there is no a pair of disjoint \( \gamma P_S \)-open sets, one containing \( x \) and the other containing \( y \).

**Example 3.8.** Consider the space \( X = \{a, b, c, d\} \) with the topology \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \). Define an operation \( \gamma \) on \( \tau \) by \( \gamma(A) = A \) for all \( A \in \tau \). Thus, \( \tau_\gamma = \tau, \tau_\gamma P_S O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \) and \( \tau_\gamma P_S GC(X) = \{\emptyset, X, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\} \).

So the space \((X, \tau)\) is \( \gamma P_S T_{1 \frac{1}{2}} \), but it is not \( \gamma P_S T_1 \) since for the points \( a \) and \( d \) in \( X \), there is no \( \gamma P_S \)-open set containing \( d \) but not \( a \).
Example 3.9. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$. Define an operation $\gamma: \tau \rightarrow P(X)$ as follows: for every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } c \in A \\ Cl(A) & \text{if } c \notin A \end{cases}$$

Clearly, $\tau_\gamma = \{\phi, X, \{c\}, \{b, c\}, \{a, b\}\}$, $\tau_\gamma - P_S O(X) = \{\phi, X, \{c\}, \{a, b\}, \{a, c\}\}$ and $\tau_\gamma - P_S GC(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$. Then $(X, \tau)$ is $\gamma - P_S - T_0$ but it is not $\gamma - P_S - T_1$, since the set $\{b, c\}$ is $\gamma - P_S - g$-closed, but it is not $\gamma - P_S$-closed.

More relations between the $\gamma - P_S - T_i$ with other types of $\gamma - pre T_i$ for $i = 0, 1, 2$ are provided as follows:

Lemma 3.10. Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. If $X$ is $\gamma - P_S - T_i$, then $X$ is $\gamma - pre T_i$ for $i = 0, 1, 2$.

Proof. The proof is obvious since every $\gamma - P_S$-open set is $\gamma$-preopen.

The converse of the Lemma 3.10 may not be true as seen in the following example.

Example 3.11. Consider the space $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Define an operation $\gamma: \tau \rightarrow P(X)$ by $\gamma(A) = A$ for all $A \in \tau$. Then $\tau_\gamma = \tau$. Therefore, the space $(X, \tau)$ is $\gamma$-pre$T_i$ but it is not $\gamma - P_S - T_i$ for $i = 0, 1, 2$.

Corollary 3.12. Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. If $X$ is $\gamma - P_S - T_i$, then $X$ is $\gamma - \beta T_i$ for $i = 0, 1, 2$.

Theorem 3.13. [2] Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. If $(X, \tau)$ is $\gamma$-semi$T_1$, then the notion of $\gamma$-P$S$-open set and $\gamma$-preopen set are identical, or this means that the notion of $\gamma$-P$S$-closed set and $\gamma$-preclosed set are identical.

Theorem 3.14. [2] Let $(X, \tau)$ be $\gamma$-semi$T_1$ space and $\gamma$ be an operation on $\tau$. A set $A$ in $X$ is $\gamma$-P$S$-$g$-closed if and only if $A$ is $\gamma$-preg-closed.

Notice that the converse of Lemma 3.10 is true when a space $X$ is $\gamma$-semi$T_1$ as shown in the following.

Corollary 3.15. Let $(X, \tau)$ be a $\gamma$-semi$T_1$ space. Then $(X, \tau)$ is $\gamma$-P$S$-$T_i$ if and only if $(X, \tau)$ is $\gamma$-pre$T_i$ for $i = 0, 1, 2$.

Proof. Directly follows from Theorem 3.13 and Theorem 3.14.

The following theorem is a relation between the $\gamma$-$P_S$-$T_i$ with other types of $\gamma$-semi$T_i$ for $i = 0, 1$. 

Theorem 3.16. Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). If \(X\) is \(\gamma-P_S-T_i\), then \(X\) is \(\gamma\)-semi\(T_i\) for \(i = 0, 1\).

Proof. (1) For \(i = 0\), let \(X\) be a \(\gamma-P_S-T_0\) space and \(x\) and \(y\) be any two distinct points of \(X\). Then there exists a \(\gamma-P_S\)-open set \(G\) containing \(x\) or \(y\) (say, \(x\) but not \(y\)). Thus, by Definition 2.3, there exists a \(\gamma\)-semiclosed set \(F\) such that \(x \in F \subseteq G\). So \(X\setminus F\) is a \(\gamma\)-semiopen set containing \(y\), and it is obvious that \(x \notin X\setminus F\). Therefore, \(X\) is \(\gamma\)-semi\(T_0\) space.

(2) For \(i = 1\), then the proof is similar to the part (1). \(\square\)

Definitions 2.3 and 3.1 lead us to the following remark.

Remark 3.17. Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\), then the following are true:

1. If \(X\) is \(\gamma-P_S-T_0\), then for each pair of distinct points \(x, y\) in \(X\), there exists a \(\gamma\)-semiclosed set containing one, but not the other.

2. If \(X\) is \(\gamma-P_S-T_1\), then for each pair of distinct points \(x, y\) in \(X\), there exist two \(\gamma\)-semiclosed sets one containing \(x\) but not \(y\), and the other containing \(y\) but not \(x\).

3. If \(X\) is \(\gamma-P_S-T_2\), then for each pair of distinct points \(x, y\) in \(X\), there exist two disjoint \(\gamma\)-semiclosed sets one containing \(x\) and the other containing \(y\).

Theorem 3.18. Let \(\gamma\) be an operation on \(\tau\), then:

1. Every topological space \((X, \tau)\) is \(\gamma\)-pre\(T_{\frac{1}{2}}\) [6].

2. Every topological space \((X, \tau)\) is \(\gamma\)-\(T_{\frac{1}{2}}\) [7].

From Lemma 3.6, Lemma 3.10, Corollary 3.12, Theorem 3.16 and Theorem
3.18. The Figure 1 is illustrated.

\[
\begin{array}{c}
\gamma^{-P_S}T_2 \rightarrow \gamma^{-P_S}T_1 \rightarrow \gamma^{-P_S}T_2 \rightarrow \gamma^{-P_S}T_0 \\
\gamma^{-pre}T_2 \rightarrow \gamma^{-pre}T_1 \rightarrow \gamma^{-pre}T_2 \leftarrow \gamma^{-pre}T_0 \\
\gamma^{-\beta}T_2 \rightarrow \gamma^{-\beta}T_1 \leftarrow \gamma^{-\beta}T_2 \rightarrow \gamma^{-\beta}T_0 \\
\gamma^{-semi}T_2 \rightarrow \gamma^{-semi}T_1 \rightarrow \gamma^{-semi}T_1 \leftarrow \gamma^{-semi}T_0 \\
\gamma^{-P_S}T_2 \rightarrow \gamma^{-P_S}T_1 \rightarrow \gamma^{-P_S}T_2 \rightarrow \gamma^{-P_S}T_0
\end{array}
\]

Figure 1

It is notice from Figure 1 that if a topological space \((X, \tau)\) is \(\gamma^{-P_S}T_0\), then \(X\) is \(\gamma^{-pre}T_2\) and hence it is \(\gamma^{-\beta}T_2\). Also, if \(X\) is \(\gamma^{-semi}T_0\), then \(X\) is both \(\gamma^{-pre}T_2\) and \(\gamma^{-\beta}T_2\). Moreover, there is no relation between \(\gamma^{-P_S}T_2\) space and \(\gamma^{-semi}T_2\) space. Also, the spaces \(\gamma^{-P_S}T_2\) and \(\gamma^{-semi}T_2\) are independent. Finally, the spaces \(\gamma^{-pre}T_i\) and \(\gamma^{-semi}T_i\) for \(i = 1, 2\) are independent.

**Definition 3.19.** Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and \(\gamma\) be an operation on \(\tau\). A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is called \(\gamma^{-P_S}\)-continuous [3] (respectively, \((\gamma, \beta)-P_S\)-irresolute [5]) if the inverse image of every open (respectively, \(\beta^{-P_S}\)-open) set in \(Y\) is \(\gamma^{-P_S}\)-open (respectively, \(\gamma^{-P_S}\)-open) set in \(X\).

**Theorem 3.20.** A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) with \(\gamma\) be an operation on \(\tau\) is \(\gamma^{-P_S}\)-continuous [3] (respectively, \((\gamma, \beta)-P_S\)-irresolute [5]) if and only if \(f^{-1}(V)\) is \(\gamma^{-P_S}\)-open set in \(X\), for every open (respectively, \(\beta^{-P_S}\)-open) set \(V\) of \(Y\).

**Definition 3.21.** A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is called \(\beta^{-P_S}\)-open [3] (respectively, \((\gamma, \beta)-P_S\)-open [5] and \(\beta\)-open [3]) if the image of every open (respectively, \(\beta^{-P_S}\)-open and open) set in \(X\) is \(\gamma^{-P_S}\)-open (respectively, \(\gamma^{-P_S}\)-open and open) set in \(Y\).
Theorem 3.22. Let $\gamma$ and $\beta$ be operations on $\tau$ and $\sigma$ respectively. Let $f: (X, \tau) \to (Y, \sigma)$ be an injective $(\gamma, \beta)$-$P_S$-irresolute function. If $(Y, \sigma)$ is $\beta$-$P_S$-$T_2$, then $(X, \tau)$ is $\gamma$-$P_S$-$T_2$.

Proof. Let $x_1$ and $x_2$ be any distinct points of a space $(X, \tau)$. Since $f$ is an injective function and $(Y, \sigma)$ is $\beta$-$P_S$-$T_2$. Then there exist two $\beta$-$P_S$-open sets $G_1$ and $G_2$ in $Y$ such that $f(x_1) \in G_1$, $f(x_2) \in G_2$ and $G_1 \cap G_2 = \emptyset$. Since $f$ is $(\gamma, \beta)$-$P_S$-irresolute, then by Theorem 3.20, $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are $\gamma$-$P_S$-open sets in $(X, \tau)$ containing $x_1$ and $x_2$ respectively. Hence $f^{-1}(G_1) \cap f^{-1}(G_2) = \emptyset$. Therefore, $(X, \tau)$ is $\gamma$-$P_S$-$T_2$.

Corollary 3.23. Let $f: (X, \tau) \to (Y, \sigma)$ be an injective $(\gamma, \beta)$-$P_S$-irresolute function. If $(Y, \sigma)$ is $\beta$-$P_S$-$T_i$, then $(X, \tau)$ is $\gamma$-$P_S$-$T_i$ for $i = 0, 1$.

Proof. The proof is similar to Theorem 3.22.

Definition 3.24. [6] A function $f: (X, \tau) \to (Y, \sigma)$ is called $\gamma$-continuous if $f^{-1}(V)$ is $\gamma$-open set in $X$, for every open set $V$ of $Y$.

Lemma 3.25. [5] Let $\gamma$ and $\beta$ be operations on $\tau$ and $\sigma$ respectively. If $f: (X, \tau) \to (Y, \sigma)$ is both $\gamma$-continuous and $\beta$-open function, then $f$ is $(\gamma, \beta)$-$P_S$-irresolute.

Lemma 3.26. Let $f: (X, \tau) \to (Y, \sigma)$ be an injective $\gamma$-continuous and $\beta$-open function. If $(Y, \sigma)$ is $\beta$-$P_S$-$T_i$, then $(X, \tau)$ is $\gamma$-$P_S$-$T_i$ for $i = 0, 1, 2$.

Proof. Directly follows from Theorem 3.22, Corollary 3.23 and Lemma 3.25 since every $\gamma$-continuous and $\beta$-open function $f$ is $(\gamma, \beta)$-$P_S$-irresolute.

Lemma 3.27. Let $\gamma$ and $\beta$ be operations on $\tau$ and $\sigma$ respectively. Let $f: (X, \tau) \to (Y, \sigma)$ be an injective $\gamma$-$P_S$-continuous function. If $(Y, \sigma)$ is $T_i$, then $(X, \tau)$ is $\gamma$-$P_S$-$T_i$ for $i = 0, 1, 2$.

Proof. It is enough to proof for one case of $i$ (say $i = 2$) since the proofs of the other cases are similar.

Let $f$ be an injective $\gamma$-$P_S$-continuous function and $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since $(Y, \sigma)$ is $T_2$, then there exist two open sets $G_1$ and $G_2$ in $(Y, \sigma)$ such that $f(x_1) \in G_1$, $f(x_2) \in G_2$ and $G_1 \cap G_2 = \emptyset$. Since $f$ is $\gamma$-$P_S$-continuous, then by Theorem 3.20, $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are $\gamma$-$P_S$-open sets in $(X, \tau)$ containing $x_1$ and $x_2$ respectively. Hence $f^{-1}(G_1) \cap f^{-1}(G_2) = \emptyset$. So $(X, \tau)$ is $\gamma$-$P_S$-$T_2$. \qed
Theorem 3.28. Assume that a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a surjective and \((\gamma, \beta)-P_S\)-open. If \((X, \tau)\) is \(\gamma-P_S-T_i\), then \((Y, \sigma)\) is \(\beta-P_S-T_i\) for \(i = 0, 1, 2\).

Proof. It is enough to prove for one case of \(i\) (say \(i = 2\)) since the proofs of the other cases are similar.

Let a function \(f\) be a surjective and \((\gamma, \beta)-P_S\)-open and \(y_1, y_2 \in Y\) such that \(y_1 \neq y_2\). Then there exist distinct points \(x_1, x_2 \in X\) such that \(f(x_1) = y_1\) and \(f(x_2) = y_2\). Since \((X, \tau)\) is \(\gamma-P_S-T_2\) space, there exist \(\gamma-P_S\)-open sets \(V_1\) and \(V_2\) such that \(x_1 \in V_1, x_2 \in V_2\) and \(V_1 \cap V_2 = \emptyset\). Since \(f\) is \((\gamma, \beta)-P_S\)-open, then \(f(V_1)\) and \(f(V_2)\) are \(\beta-P_S\)-open sets in \((Y, \sigma)\) such that \(y_1 = f(x_1) \in f(V_1)\) and \(y_2 = f(x_2) \in f(V_2)\). This implies that \(f(V_1) \cap f(V_2) = \emptyset\). Hence \((Y, \sigma)\) is \(\beta-P_S-T_2\).

The proof of the next corollary is similar to Theorem 3.28.

Corollary 3.29. Assume that a function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is a surjective and \(\beta-P_S\)-open. If \((X, \tau)\) is \(T_i\), then \((Y, \sigma)\) is \(\beta-P_S-T_i\) for \(i = 0, 1, 2\).

Proof. Obvious.

The following new space in terms of \(\tau_\gamma P_S\)-closure will help us to give more relations and properties of \(\gamma-P_S-T_i\) spaces for \(i = 0, \frac{1}{2}, 1, 2\).

Definition 3.30. A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\), is said to be \(\gamma-P_S\)-symmetric if for each \(x, y \in X\), then \(x \in \tau_\gamma P_S Cl(\{y\})\) implies that \(y \in \tau_\gamma P_S Cl(\{x\})\).

Theorem 3.31. Let \((X, \tau)\) be a topological space and \(\gamma\) be an operation on \(\tau\). Then \(X\) is \(\gamma-P_S\)-symmetric if and only if the singleton set \(\{x\}\) is \(\gamma-P_S\)-g-closed, for each \(x \in X\).

Proof. Let \(\{x\} \subseteq G\) and \(G\) is \(\gamma-P_S\)-open set in \(X\). Suppose that \(\tau_\gamma P_S Cl(\{x\}) \not\subseteq G\). Then \(\tau_\gamma P_S Cl(\{x\}) \cap X \setminus G \neq \emptyset\). Let \(y \in \tau_\gamma P_S Cl(\{x\}) \cap X \setminus G\), then \(y \in \tau_\gamma P_S Cl(\{x\})\). Since a space \(X\) is \(\gamma-P_S\)-symmetric, then \(x \in \tau_\gamma P_S Cl(\{y\}) \subseteq X \setminus G\) and hence \(x \notin G\). This is a contradiction of the assumption. This means that the singleton set \(\{x\}\) is \(\gamma-P_S\)-g-closed, for each \(x \in X\).

Conversely, suppose that \(x \in \tau_\gamma P_S Cl(\{y\})\), but \(y \notin \tau_\gamma P_S Cl(\{x\})\). Then \(\{y\} \subseteq X \setminus \tau_\gamma P_S Cl(\{x\})\) and \(X \setminus \tau_\gamma P_S Cl(\{x\})\) is \(\gamma-P_S\)-open set in \(X\). Then by hypothesis, we have \(\tau_\gamma P_S Cl(\{y\}) \subseteq X \setminus \tau_\gamma P_S Cl(\{x\})\). Therefore \(x \in X \setminus \tau_\gamma P_S Cl(\{x\})\). This is contradiction. Therefore, \(y \in \tau_\gamma P_S Cl(\{x\})\) and hence \(X\) is \(\gamma-P_S\)-symmetric space.
Corollary 3.32. [4] In a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$. Then every subset of $X$ is $\gamma$-$P_S$-g-closed if and only if $\tau_\gamma P_S O(X) = \tau_\gamma P_S C(X)$.

Lemma 3.33. For any topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$. If $\tau_\gamma P_S O(X) = \tau_\gamma P_S C(X)$, then $(X, \tau)$ is $\gamma$-$P_S$-symmetric.

Proof. Since $\tau_\gamma P_S O(X) = \tau_\gamma P_S C(X)$, then by Corollary 3.32, every subset of $(X, \tau)$ is $\gamma$-$P_S$-g-closed and hence by Theorem 3.31, the space $(X, \tau)$ is $\gamma$-$P_S$-symmetric. 

Recall that a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is $\gamma$-locally indiscrete if every $\gamma$-open subset of $X$ is $\gamma$-closed, or every $\gamma$-closed subset of $X$ is $\gamma$-open [1].

Theorem 3.34. [4] If a space $(X, \tau)$ is $\gamma$-locally indiscrete, then $\tau_\gamma P_S O(X) = \tau_\gamma P_S C(X)$.

From Lemma 3.33 and Theorem 3.34, we have the following lemma.

Lemma 3.35. If $(X, \tau)$ is $\gamma$-locally indiscrete space, then $(X, \tau)$ is $\gamma$-$P_S$-symmetric.

Proof. Clear.

Remark 3.36. If $(X, \tau)$ is $\gamma$-locally indiscrete space such that $\tau_\gamma \neq P(X)$, then $(X, \tau)$ will be not $\gamma$-$P_S$-$T_i$ for $i = 0, 1, 2$.

Recall that a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is $\gamma$-hyperconnected if $\tau_\gamma Cl(G) = X$ for every $\gamma$-open set $G$ of $X$ [1].

Theorem 3.37. [2] Let $\gamma$ be a regular operation on $\tau$. A topological space $(X, \tau)$ is $\gamma$-hyperconnected if and only if $\tau_\gamma P_S O(X) = \{\emptyset, X\}$.

Corollary 3.38. If $(X, \tau)$ is $\gamma$-hyperconnected space and $\gamma$ is a regular operation on $\tau$, then $(X, \tau)$ will be not $\gamma$-$P_S$-$T_i$ for $i = 0, 1, 2$. This means that if $(X, \tau)$ is $\gamma$-$P_S$-$T_i$ for $i = 0, 1, 2$, then it is not $\gamma$-hyperconnected.

Proof. This is an immediate consequence of Theorem 3.37.

Lemma 3.39. If $(X, \tau)$ is $\gamma$-hyperconnected space and $\gamma$ is a regular operation on $\tau$, then $(X, \tau)$ is $\gamma$-$P_S$-symmetric.

Proof. Directly follows from Lemma 3.33 and Theorem 3.37.
The relation between $\gamma$-$P_S$-symmetric and $\gamma$-$P_S$-$T_1$ spaces are shown in the following theorem.

**Theorem 3.40.** If $(X, \tau)$ is $\gamma$-$P_S$-$T_1$ space, then it is $\gamma$-$P_S$-symmetric.

**Proof.** By Theorem 3.3, every singleton sets are $\gamma$-$P_S$-closed in a $\gamma$-$P_S$-$T_1$ space $(X, \tau)$. Since every $\gamma$-$P_S$-closed set is $\gamma$-$P_S$-$g$-closed. Then by Theorem 3.31, $(X, \tau)$ is $\gamma$-$P_S$-symmetric.

But the converse of Theorem 3.40 may not be true as in the next example shows.

**Example 3.41.** Let $(X, \tau)$ be any $\gamma$-hyperconnected space and $\gamma$ be a regular operation on $\tau$, then by Lemma 3.39, the space $(X, \tau)$ is $\gamma$-$P_S$-symmetric, but it is not $\gamma$-$P_S$-$T_1$ since $X$ is $\gamma$-hyperconnected space and hence by Theorem 3.37, $\tau_\gamma P_S O(X) = \{\phi, X\}$.

Since every $\gamma$-$P_S$-$T_2$ space is $\gamma$-$P_S$-$T_1$, then by Theorem 3.40 that every $\gamma$-$P_S$-$T_2$ space is $\gamma$-$P_S$-symmetric. But there is no relation between $\gamma$-$P_S$-$T_0$ and $\gamma$-$P_S$-symmetric spaces.

**Theorem 3.42.** Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. Then $(X, \tau)$ is $\gamma$-$P_S$-$T_1$ if and only if $(X, \tau)$ is $\gamma$-$P_S$-$T_0$ and $\gamma$-$P_S$-symmetric.

**Proof.** If $(X, \tau)$ is $\gamma$-$P_S$-$T_1$, then by Lemma 3.6 and Theorem 3.40, $(X, \tau)$ is both $\gamma$-$P_S$-$T_0$ and $\gamma$-$P_S$-symmetric respectively.

Conversely, let $x, y$ be any two distinct points of $\gamma$-$P_S$-$T_0$ space $(X, \tau)$, then by hypothesis there exists a $\gamma$-$P_S$-open set $G$ containing $x$ or $y$ (say $x$, but not $y$). This means that $x \in G \subseteq X \setminus \{y\}$. Then $G \cap \{y\} = \phi$ and hence $x \notin \tau_\gamma P_S Cl(\{y\})$. Since $(X, \tau)$ is $\gamma$-$P_S$-symmetric, then $y \notin \tau_\gamma P_S Cl(\{x\})$ which implies that there exists a $\gamma$-$P_S$-open set $H$ such that $y \in H \subseteq X \setminus \{x\}$. Therefore, the space $(X, \tau)$ is $\gamma$-$P_S$-$T_1$.

**Corollary 3.43.** Let $(X, \tau)$ be any $\gamma$-$P_S$-symmetric space and $\gamma$ be an operation on $\tau$. Then the following properties are equivalent:

1. $X$ is $\gamma$-$P_S$-$T_1$.
2. $X$ is $\gamma$-$P_S$-$T_{1/2}$.
3. $X$ is $\gamma$-$P_S$-$T_0$.

**Proof.** The proof of the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are follows directly from Lemma 3.6.
(3) \Rightarrow (1) It is clear from Theorem 3.42 since \((X, \tau)\) is \(\gamma\)-\(P_S\)-symmetric space.

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References


