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NEW SELF-STARTING APPROACH FOR SOLVING SPECIAL THIRD ORDER INITIAL VALUE PROBLEMS

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Abstract: Third order initial value problems are generally solved numerically by developing numerical methods designed for third order differential equations. Also, to improve the accuracy of the methods, most of the numerical methods are implemented with starting values. However, it has been observed that a third derivative method developed for solving second order initial value problems can also be adopted to solve third order initial value problems. Although, this third derivative method is adopted in self-starting mode, it is seen to be very relevant from the results obtained in this paper. This third derivative block method is adopted to solve some special third order numerical problems previously solved in literature and the method gave better results despite that the previous methods have equal and higher order.

AMS Subject Classification: 65L05, 65L06
Key Words: third-derivative, self-starting, block method, special third order, initial value problems

1. Introduction

This paper considers the special third order initial value problem of the form

$$y''' = f\left(x, y\right), y(a) = \alpha, y'(a) = \beta, y''(a) = \gamma \tag{1}$$

where α, β and γ are given constants.

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The direct solution of the general form of (1) numerically without reducing to a system of first order initial value problems have been considered by several studies such as [4, 3, 1, 14, 12, 7, 5] amongst others.

Some studies who have considered the solution of (1) being in special form includes [11, 10, 11, 2]. [11, 10] developed a five-step and six-step method respectively for the solution of (1). Also, [2] adopted an hybrid block method while [11] used a three-step block method. Despite the variations in the methods adopted, the introduction of higher derivative was not considered.

Although, the introduction of higher derivative which dates back to [8] have been considered by [6] and [13], the studies only considered the methods to solve the order the methods were developed for and not an order higher. That is, [6] developed a second order third derivative method (TDM) to solve corresponding second order initial and boundary value problems and also, [13] developed a third order fourth derivative method (FDM) to solve third order boundary value problems.

Hence, this paper focuses on the application of a second order block method being applied to solve special third order initial value problems. A third derivative is introduced in the second order block method which aids in the application to third order initial value problems. In the implementation, it is stated that the block was implemented without starting values (self-starting) and then applied to solve some special third order numerical problems to show the improvement in accuracy.

This paper is organized as follows. In Section Two, the derivation of the third derivative block method is shown while Section Three gives two numerical examples to compare the methods for accuracy. Finally, the conclusion of the paper is given in Section Four.

2. Derivation of the Method

This section shows the development of the third derivative method with step-number $k = 2$. The discrete scheme for the block method is constructed from the linear multistep method form given below

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j} \tag{2}$$

where

$$f_{n+j} = f\left(x_{n+j}, y_{n+j}, y'_{n+j}\right), \quad g_{n+j} = \frac{df\left(x, y(x), y'(x)\right)}{dx},$$

$$g_{n+j} = g\left(x_{n+j}, y_{n+j}, y'_{n+j}\right).$$

Using Taylor series expansion to expand individual terms in (2) and upon substitution of the expansions back in (2), the matrix form can be written as below where the coefficients of $y^{(m)}x_n$ are equated

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(h)^2}{2!} & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{(h)^3}{3!} & 0 & h & 2h & 1 & 1 & 1 \\ 0 & \frac{(h)^4}{4!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & 0 & h & 2h \\ 0 & \frac{(h)^5}{5!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} \\ 0 & \frac{(h)^6}{6!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} \\ 0 & \frac{(h)^7}{7!} & 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2h \\ \frac{(2h)^2}{2!} \\ \frac{(2h)^3}{3!} \\ \frac{(2h)^4}{4!} \\ \frac{(2h)^5}{5!} \\ \frac{(2h)^6}{6!} \\ \frac{(2h)^7}{7!} \end{pmatrix}$$

The values of $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \lambda_0, \lambda_1$ and λ_2 are obtained using matrix inverse method as given below

$$(\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \lambda_0, \lambda_1, \lambda_2)^T = \left(-1, 2, \frac{2h^2}{15}, \frac{11h^2}{15}, \frac{2h^2}{15}, \frac{h^3}{40}, 0, -\frac{h^3}{40}\right)^T \tag{3}$$

Substituting (3) in (2) gives the discrete scheme

$$y_{n+2} = -y_n + 2y_{n+1} + \frac{h^2}{15}\left(2f_n + 11f_{n+1} + 2f_{n+2}\right) + \frac{h^3}{40}\left(g_n - g_{n+2}\right) \tag{4}$$

Adopting the same approach, the following derivatives of the discrete scheme are obtained

$$\begin{aligned} y'_n &= \tfrac{1}{h}\left(-y_n + y_{n+1}\right) + \tfrac{h}{42}\left(-13f_n - 7f_{n+1} - f_{n+2}\right) \\ &\quad + \tfrac{h^2}{1680}\left(-59g_n + 128g_{n+1} + 11g_{n+2}\right), \\ y'_{n+1} &= \tfrac{1}{h}\left(-y_n + y_{n+1}\right) + \tfrac{h}{1680}\left(187f_n + 616f_{n+1} + 37f_{n+2}\right) \\ &\quad + \tfrac{h^2}{840}\left(16g_n - 76g_{n+1} - 5g_{n+2}\right), \\ y'_{n+2} &= \tfrac{1}{h}\left(-y_n + y_{n+1}\right) + \tfrac{h}{70}\left(11f_n + 63f_{n+1} + 31f_{n+2}\right) \\ &\quad + \tfrac{h^2}{1680}\left(53g_n + 128g_{n+1} - 10g_{n+2}\right). \end{aligned} \tag{5}$$

Combining equations (4) and (5) in matrix form yields

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 1 & 0 \\ -\frac{1}{h} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y'_{n+1} \\ y'_{n+2} \end{pmatrix} = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} \tag{6}$$

where

$$\begin{aligned}
M_1 &= -y_n + \tfrac{h^2}{15}\left(2f_n + 11f_{n+1} + 2f_{n+2}\right) + \tfrac{h^3}{40}\left(g_n - g_{n+2}\right)\\
M_2 &= -y'_n - \tfrac{1}{h}y_n + \tfrac{h}{42}\left(-13f_n - 7f_{n+1} - f_{n+2}\right)\\
&\quad + \tfrac{h^2}{1680}\left(-59g_n + 128g_{n+1} + 11g_{n+2}\right)\\
M_3 &= -\tfrac{1}{h}y_n + \tfrac{h}{1680}\left(187f_n + 616f_{n+1} + 37f_{n+2}\right)\\
&\quad + \tfrac{h^2}{840}\left(16g_n - 76g_{n+1} - 5g_{n+2}\right)\\
M_4 &= -\tfrac{1}{h}y_n + \tfrac{h}{70}\left(11f_n + 63f_{n+1} + 31f_{n+2}\right)\\
&\quad + \tfrac{h^2}{1680}\left(53g_n + 128g_{n+1} - 10g_{n+2}\right)
\end{aligned}$$

Adopting matrix inverse method, $y_{n+1}, y_{n+2}, y'_{n+1}$ and y'_{n+2} are determined and expressed as given below

$$\begin{aligned}
y_{n+1} &= y_n + hy'_n + \tfrac{h^2}{42}\left(13f_n + 7f_{n+1} + f_{n+2}\right)\\
&\quad + \tfrac{h^3}{1680}\left(59g_n - 128g_{n+1} - 11g_{n+2}\right),\\
y_{n+2} &= y_n + 2hy'_n + \tfrac{h^2}{105}\left(79f_n + 112f_{n+1} + 19f_{n+2}\right)\\
&\quad + \tfrac{h^3}{105}\left(10g_n - 16g_{n+1} - 4g_{n+2}\right),\\
y'_{n+1} &= y'_n + \tfrac{h}{240}\left(101f_n + 128f_{n+1} + 11f_{n+2}\right)\\
&\quad + \tfrac{h^2}{240}\left(13g_n - 40g_{n+1} - 3g_{n+2}\right),\\
y'_{n+2} &= y'_n + \tfrac{h}{15}\left(7f_n + 16f_{n+1} + 7f_{n+2}\right)\\
&\quad + \tfrac{h^2}{15}\left(g_n - g_{n+2}\right).
\end{aligned} \tag{7}$$

The block method in equation (7) above is a zero-stable block integrator of order six with $C_8 = \left(\frac{1}{17280}, \frac{1}{4725}, \frac{1}{9450}, \frac{1}{4725}\right)^T$

3. Numerical Problems

Problem 1.

$$\begin{aligned}
&y''' = -y, \quad y(0) = 1, y'(0) = 1, y''(0) = 1\\
&\text{Exact Solution:} \quad y(x) = e^{-x}
\end{aligned}$$

The results as displayed in Table 1 is compared with the order six method from [12] and the notation 2STDM represents the Two-Step Third Derivative Method.

Table 1: Comparison of results for solving Problem 1 ($h = 0.1$)

x	Exact	2STDM	Error [12]	Error (2STDM)
0.1	0.9048374180359596	0.9048374180354298	2.1824E-12	5.2981E-13
0.2	0.8187307530779819	0.8187307530760476	4.0342E-12	1.9342E-12
0.3	0.7408182206817179	0.7408182206774051	5.3888E-12	4.3127E-12
0.4	0.6703200460356393	0.6703200460281931	6.4744E-11	7.4462E-12
0.5	0.6065306597126334	0.6065306597011971	5.9078E-11	1.1436E-11
0.6	0.5488116360940264	0.5488116360779025	1.1672E-11	1.6124E-11
0.7	0.4965853037914095	0.4965853037697958	2.2523E-11	2.1614E-11
0.8	0.4493289641172216	0.4493289640894242	3.9121E-11	2.7797E-11
0.9	0.4065696597405991	0.4065696597058145	6.1177E-11	3.4785E-11
1.0	0.3678794411714423	0.3678794411289312	4.9220E-11	4.2511E-11

Problem 2.

$$y''' = 3\sin(x), \quad y(0) = 1, y'(0) = 0, y''(0) = -2$$
$$\text{Exact Solution:} \quad y(x) = 3\cos(x) + \tfrac{x^2}{2} - 2$$

The results as displayed in Table 2 is compared with the order six method from [2] and the notation 2STDM represents the Two-Step Third Derivative Method.

Table 2: Comparison of results for solving Problem 2 ($h = 0.1$)

x	Exact	2STDM	Error [2]	Error (2STDM)
0.1	0.990012495834077	0.9900124958323491	2.5934E-12	1.7282E-12
0.2	0.960199733523725	0.9601997335174070	1.1857E-11	6.3179E-12
0.3	0.911009467376818	0.9110094673625230	2.6224E-11	1.4295E-11
0.4	0.843182982008655	0.8431829819836356	4.7034E-11	2.5020E-11
0.5	0.757747685671118	0.7577476856321904	7.2700E-11	3.8928E-11
0.6	0.656006844729035	0.6560068446736752	1.0437E-10	5.5360E-11
0.7	0.539526561853465	0.5395265617788210	1.4049E-10	7.4644E-11
0.8	0.410120128041496	0.4101201279453678	1.8197E-10	9.6128E-11
0.9	0.269829904811993	0.2698299046919727	2.2736E-10	1.2002E-10
1.0	0.120906917604419	0.1209069174587185	2.7729E-10	1.4570E-10

Problem 3.

$$y''' = e^x, \quad y(0) = 3, y'(0) = 1, y''(0) = 5$$
$$\text{Exact Solution:} \quad y(x) = 2 + 2x^2 + e^x$$

The results as displayed in Table 3 is compared with the order eight method from [10] and the notation 2STDM represents the Two-Step Third Derivative Method.

Table 3: Comparison of results for solving Problem 3 ($h = 0.1$)

x	Exact	2STDM	Error [10]	Error (2STDM)
0.1	3.125170918075647	3.125170918075013	9.24352E-10	6.34270E-13
0.2	3.301402758160170	3.301402758157841	18.3983E-10	2.32882E-12
0.3	3.529858807576003	3.529858807570560	24.2400E-10	5.44348E-12
0.4	3.811824697641270	3.811824697631417	53.5873E-10	9.85317E-12
0.5	4.148721270700128	4.148721270684131	7.00128E-10	1.59974E-11
0.6	4.542118800390509	4.542118800366786	3.90509E-10	2.37223E-11
0.7	4.993752707470477	4.993752707436909	6.52952E-09	3.35679E-11
0.8	5.505540928492468	5.505540928447123	2.15075E-08	4.53443E-11
0.9	6.079603111156950	6.079603111097241	3.88430E-08	5.97084E-11
1.0	6.718281828459045	6.718281828382613	6.15410E-08	7.64322E-11

4. Conclusion

This paper has presented a new approach for solving special third order initial value problems using a second order third derivative block method. The introduction of higher derivative in the block method gave room for a second order block method to adequately solve a third order initial value problem. Also despite that the method was implemented without starting values and compared with methods of higher order of accuracy, the third derivative method still displayed better results as shown in the tables above. Hence, it is clear that the approach is an adequate one for solving special third order initial value problems.

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