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Estimation of Information Measures for Power-Function Distribution in Presence of Outliers and Their Applications

[1]Amal Soliman Hassan, [2]Elsayed Ahmed Elsherpieny & [*3]Rokaya Elmorsy Mohamed
[1,2&3]Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt
[3]Department of Mathematics, Statistics and Insurance, Sadat Academy for Management Sciences, Egypt

[1]amal52_soliman, [2]Elsayed.Elsherpieny@cu.edu.eg
[3]Rokaya.Elmorsy@sadatacademy.edu.eg
[*]Corresponding author

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ABSTRACT

Entropy measurement plays an important role in the field of information theory. Furthermore, the estimation of entropy is an important issue in statistics and machine learning. This study estimated the Rényi and q-entropies of a power-function distribution in the presence of s outliers using classical and Bayesian procedures. In the classical method, the maximum likelihood estimators of the entropies were obtained and their performance was assessed through a numerical study. In the Bayesian method, the Bayesian estimators of the entropies under uniform and gamma priors were acquired based on different loss functions. The Bayesian estimators were computed empirically using a Monte Carlo simulation based on the Gibbs sampling algorithm. The simulated datasets were analyzed to investigate the accuracy of the estimates. The study results showed that the precision of the maximum likelihood and Bayesian estimates of both entropies improved with increasing the sample size and the number of outliers. The absolute biases and the mean squared errors of the estimates in the presence of outliers exceeded those of the corresponding estimates in the homogenous case (no-outliers). Furthermore, the Bayesian estimates of the Rényi and *q*-entropies under the squared error loss function were preferable to the other Bayesian estimates in a majority of the cases. Finally, analysis results of real data examples were consistent with those of the simulated data.

Keywords: Bayesian estimators, maximum likelihood estimators, outliers, power-function distribution, Rényi entropy.

INTRODUCTION

Power-function distribution (PFD) is one of the most significant parametric models. It is usually used in the analysis of lifetime data and solving problems related to the modeling of failure processes. It is an elastic lifetime model that provides a good fit for some sets of failure data. The probability density function (PDF) and cumulative distribution function (CDF) of PFD with scale parameter θ and shape paramete α are provided respectively in Equations 1 and 2 as follows:

$$f_1(x\,;\alpha,\theta)=\frac{\alpha}{\theta^{\alpha}}x^{\alpha-1};\quad ,\alpha>0,\ 0<x<\theta, \tag{1}$$

and

$$F_1(x;\alpha,\theta)=\left(x/\theta\right)^{\alpha}. \tag{2}$$

In the literature, PFD has been discussed by several authors. Malik (1967) derived the moments of order statistics for PFD. Ahsanullah and Kabir (1974) discussed the characterizations of PFD. Rahman et al. (2012) performed the Bayesian estimation of PFD under conjugate prior. Sultan et al. (2014) discussed Bayesian estimation for PFD under double priors. The Bayesian estimator of PFD was obtained by Zaka and Akhter (2014). Abdul-Sathar and Sathyareji (2018) studied the estimation of dynamic cumulative past entropy for PFD.

Entropy is one of the significant measures in statistical mechanics. It is essentially assigned in physics, in particular, in the second law of thermodynamics. The notion of entropy was proposed by Shannon (1948) as a quantitative measure of uncertainty. This concept of entropy is axiomatic for rare events and outlier detection because the presence of outliers increases the entropy (randomness or uncertainty) of a dataset, and this increment can be used to measure the outlier of an object.

The Rényi and q-entropies are parametric extensions of the Shannon entropy introduced by Rényi (1961) and Havrda and Charvát (1967), respectively. Tsallis (1988) applied the q-entropy in physical problems. For a random variable X with a PDF $f(x)$, the Rényi and q-entropies measures are specified in Equations 3 and 4, respectively, as follows:

$$H_\tau\left(X\right)=\frac{1}{1-\tau}\log\left\{\int_{-\infty}^{\infty} f(x)^\tau dx\right\}, \tag{3}$$

where $\tau \neq 1$ and $\tau > 0,$ for $q \neq 1$ and $q > 0,$ we have

$$H_q\left(X\right)=\frac{1}{q-1}\left[1-\int_{-\infty}^{\infty} f(x)^q dx\right]. \tag{4}$$

Several studies concerning the estimation of entropy have been reported. For example, Cramer and Bagh (2011) discussed the entropy in the Weibull distribution for progressive censoring. Cho et al. (2014) derived the maximum likelihood (ML) estimators for the entropy measure of a Rayleigh distribution using doubly generalized type II hybrid censored samples. Chacko and Asha (2018) estimated the entropy for a generalized exponential distribution under record values. Liu and Gui (2019) studied the Shannon entropy for a Lomax distribution using the generalized progressively hybrid censoring scheme. Hassan and Zaky (2019) investigated the ML estimator of the Shannon entropy for an inverse Weibull distribution using multiple censored samples. Ahmadini et al. (2021) analyzed the Bayesian and credible interval estimators of the dynamic cumulative residual entropy for a Pareto II model. The entropy Bayesian estimator for a Lomax distribution was studied by Hassan and Zaky (2021) based on record values.

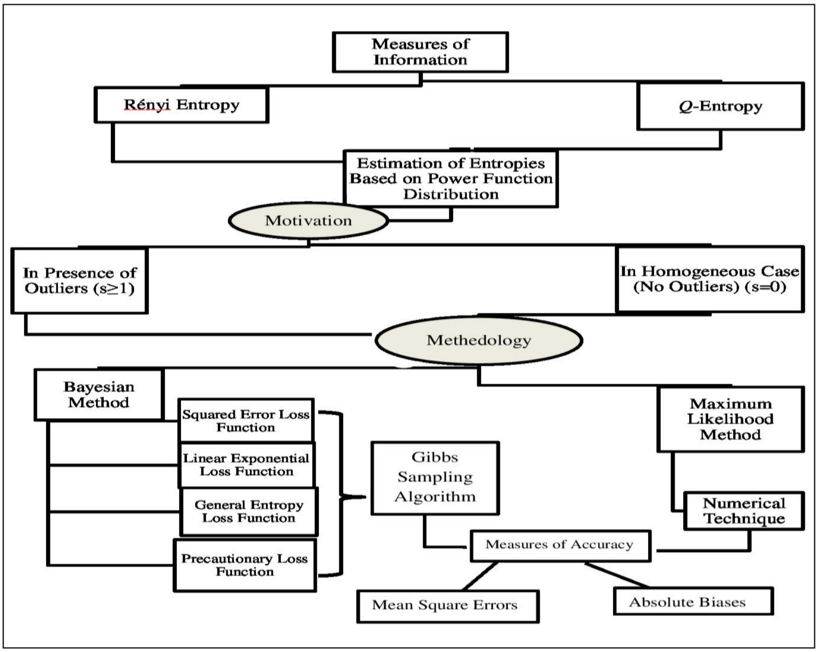
Outliers can negatively affect statistical analysis. Therefore, outliers can increase error variance and reduce the power of statistical tests.

Furthermore, outliers can seriously bias or influence estimates that may be of substantive interest for more information on these problems (Hamid, 2018a, 2018b; Kumaran et al., 2020; Rasmussen & Vicente, 1989; Schwager & Margolin, 1982).

In the literature, there are no research works on the estimation problem of entropy measure in the presence of outliers. Therefore, the main purpose of this manuscript is to scrutinize the estimation problem of the Rényi and *q*-entropies for PFD in the presence of outliers. Herein, the ML and Bayesian estimators are derived. The Bayesian estimators of the entropies under four loss functions are obtained, namely squared error loss function (SELF), linear exponential loss function (LLF), general entropy loss function (GELF), and precautionary loss function (PLF). In addition, the Markov chain Monte Carlo (MCMC) procedure using the Gibbs sampling algorithm is employed owing to the intricate forms of entropy Bayesian estimators. Furthermore, real data experiments confirm the results of the study. The motivation and methodology of the study are demonstrated in Figure 1.

Figure 1

Motivation and Procedure of the Study



JOINT DISTRIBUTION OF RANDOM VARIABLES X_1, X_2,…, X_n IN PRESENCE OF S OUTLIERS

According to Dixit (1989) and Dixit and Nasiri (2001), let the random variables X_1, X_2,…, X_n be defined such that s of them (the number of outliers) and the outliers themselves are unknown. The expected lifetime of these unknown variables is large (or small) as compared with that of the rest variables. Assume that the s random variables are PFD with parameters, α, β , and θ having the following PDF mentioned in Equation 5 as:

$$f_2(x;\alpha,\beta,\theta) = \alpha(\beta\theta)^{-\alpha} x^{\alpha-1},\ 0 < x < \beta\theta,\ \alpha > 0,\ \beta > 1,\ \theta > 0, \tag{5}$$

where θ and β are the scale parameters, α is the shape parameter, and the remaining $(n-s)$ random variables are PFD with the PDF provided in Equation 1. The likelihood function of X_1, X_2,…, X_n in the presence of s outliers is given in Equation 6 as:

$$L(\underline{x};\Xi) = \frac{1}{C(n,s)} \prod_{i=1}^{n} f_1(x_i) \times \sum_{\underline{A}} \prod_{j=1}^{s} \frac{f_2(x_{A_j})}{f_1(x_{A_j})} = \frac{\alpha^n \theta^{-n\alpha} \beta^{-s\alpha}}{C(n,s)} \prod_{i=1}^{n} x_i^{\alpha-1} \sum_{\underline{A}} \prod_{j=1}^{s} I(\beta\theta - x_{A_j}), \tag{6}$$

where $C(n,s) = \frac{(n!)}{\left[(s!)(n-s)!\right]}$, $\Xi \equiv (\alpha,\beta,\theta)$, $\sum_{\underline{A}} = \sum_{A_1=1}^{n+1-s} \sum_{A_2=(A_1+1)}^{n+2-s} \dots \sum_{A_s=(A_{s-1}+1)}^{n}$ and $I(.)$ represents the indicator function defined as; $I(x) = \begin{cases} 1 & x > 0 \\ 0 & otherwise \end{cases}$.

Thus, the marginal distribution of PFD in the presence of s outliers is as follows:

$$\begin{aligned} f(x;\Xi) &= \left(\frac{n-s}{n}\right) f_1(x;\alpha,\theta) + \left(\frac{s}{n}\right) f_2(x;\alpha,\beta,\theta) \\ &= \bar{b}\alpha\theta^{-\alpha} x^{\alpha-1} I(\theta - x) + b\alpha(\beta\theta)^{-\alpha} x^{\alpha-1} I(\beta\theta - x), \end{aligned}$$

which leads to the formula presented in Equation 7 as follows:

$$f(x;\Xi) = \alpha\theta^{-\alpha}(b\beta^{-\alpha} + \bar{b})x^{\alpha-1}, \tag{7}$$

where $b = \frac{s}{n}$, $\bar{b} = (1-b) = \frac{(n-s)}{n}$.

EXPRESSIONS OF RÉNYI AND *Q*-ENTROPIES

Here, an explicit expression of the Rényi and q-entropies for PFD in the presence of outliers is provided. Let X be a random variable following PFD. Thus, the Rényi entropy of X is produced by substituting (7) in (3) as shown in Equation 8:

$$H_\tau(X) = \frac{1}{1-\tau}\log\left[\int_{D_x}(\alpha\theta^{-\alpha})^\tau (b\beta^{-\alpha}+\bar{b})^\tau x^{\tau(\alpha-1)}dx\right] = \frac{1}{1-\tau}\log J. \quad (8)$$

To compute the Rényi entropy, J should be obtained first as presented in Equation 9:

$$\begin{aligned} J &= (\alpha\theta^{-\alpha})^\tau\left(\frac{b}{\beta^\alpha}+\bar{b}\right)^{\tau-1}\left[b\beta^{-\alpha}\int_0^{\beta\theta}x^{(\alpha-1)\tau}dx + \bar{b}\int_0^{\theta}x^{(\alpha-1)\tau}dx\right] \\ &= (\alpha\theta^{-\alpha})^\tau\left(\frac{b}{\beta^\alpha}+\bar{b}\right)^{\tau-1}\left[\frac{\theta^{\tau(\alpha-1)+1}}{\tau(\alpha-1)+1}\left(b\beta^{(\tau-1)(\alpha-1)}+\bar{b}\right)\right]. \end{aligned} \quad (9)$$

Therefore, the Rényi entropy of PFD represented in Equation 10, in the presence of outliers, is given by substituting (9) into (8) as:

$$H_\tau(X) = \left(\frac{1}{1-\tau}\right)\log\left\{\left(b\beta^{(\tau-1)(\alpha-1)}+\bar{b}\right)\left[\frac{\alpha^\tau(b\beta^{-\alpha}+\bar{b})^{\tau-1}}{\theta^{\tau-1}\left(\tau(\alpha-1)+1\right)}\right]\right\}. \quad (10)$$

Similarly, after applying (9) in (4) with $\tau = q$, the q-entropy of X is given in Equation 11 as follows:

$$H_q(X) = \left(\frac{1}{q-1}\right)\left[1-\left(b\beta^{(q-1)(\alpha-1)}+\bar{b}\right)\left\{\frac{\alpha^q(b\beta^{-\alpha}+\bar{b})^{q-1}}{\theta^{q-1}\left(q(\alpha-1)+1\right)}\right\}\right]. \quad (11)$$

Expressions (10) and (11) of the Rényi and q-entropies are the functions of parameters and outlier number. In addition, the required expression of the Rényi and q-entropies in the homogenous (no-outlier) case is resulted from Equations 10 and 11 after inserting $\beta = 1$ or $s = 0$.

ESTIMATION OF ENTROPIES IN PRESENCE OF OUTLIERS

Here, the ML and Bayesian estimators of the entropies for PFD in the presence of s outliers as well as the ML and Bayesian estimators in the homogenous case (i.e., $s = 0$ or $\beta = 1$) are obtained.

Maximum Likelihood Estimator of Entropies

The problem of deriving the ML estimators of the Rényi and q-entropies for PFD in the presence of s outliers is considered. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from PFD in the presence of outliers with likelihood function (6). The logarithmic likelihood function, denoted by, from a sample of n observations $X_1, X_2, \ldots, X_n$ is given by:

$$\ln l = n\ln\alpha - n\alpha\ln\theta - s\alpha\ln\beta - \ln[C(n,s)] + (\alpha-1)\sum_{i=1}^{n}\ln x_i + \ln\left[\sum_{\underline{A}}\prod_{j=1}^{s} I(\beta\theta - x_{A_j})\right].$$

Assuming that the parameter θ is known, the estimate of $\beta\theta$ is the sample maxima, i.e., $\beta\theta$ is $X_{(n)} = \max\{X_i\}_{i=1}^{n}$ .Therefore, the estimator of β, say $\hat{\beta}$, is obtained in Equation 12 as:

$$\hat{\beta} = X_{(n)}/\theta . \tag{12}$$

Equation 13 gives the partial derivative of the logarithmic likelihood function with respect to as follows:

$$\partial \ln l/\partial\alpha = n/\alpha - s\ln(\beta) - n\ln(\theta) + \sum_{i=1}^{n}\ln x_i . \tag{13}$$

After substituting (12) into (13) and equating by zero, the ML estimator of α, denoted by $\hat{\alpha}$,can be expressed in Equation 14 as follows:

$$\hat{\alpha} = \frac{n}{s\ln(X_{(n)}) + (n-s)\ln(\theta) - \sum_{i=1}^{n}\ln x_i}. \tag{14}$$

Based on the invariance property of the ML method, the ML estimators of $H_\tau(X)$ and $H_q(X)$, denoted by $\hat{H}_\tau(X)$ and $\hat{H}_q(X)$, are obtained by directly substituting (12) and (14) into (3) and (4), respectively. Furthermore, the ML estimators of $H_\tau(X)$ and $H_q(X)$ are obtained in the homogenous case by setting $\beta = 1$ or $s = 0$.

Entropy Bayesian Estimators

The Bayesian estimation of $H_\tau(X)$ and $H_q(X)$ for PFD in the presence of outliers using different loss functions is discussed. The parameter θ is considered to be known and the parameters β and α have uniform and gamma distributions, respectively.

Let X_1, $X_{2\,...,}$ X_n be a random sample drawn from PFD with a set of parameters Ξ, where their likelihood function is defined in Equation 6. Thus, assuming the parameter independence and following Abdul-Sathar and Krishnan (2019), the joint prior distribution, presented in Equation 15, for α and β is given:

$$\pi_1(\alpha,\beta) \propto \left(\frac{\alpha^{c-1}\, e^{-d\alpha}}{\beta} \right) \qquad ;\alpha, c, \beta, d > 0. \tag{15}$$

Combining (6) and (15), the joint posterior distribution, represented in Equation 16, takes the form:

$$\pi_1^{\bullet}\left(\alpha,\beta \mid \underline{x}\right) \propto \alpha^{c+n-1} e^{-\alpha\left[n\ln\theta + d + s\ln\beta - \sum_{i=1}^{n}\ln x_i\right] - \sum_{i=1}^{n}\ln x_i - \ln\beta}. \tag{16}$$

The marginal posterior distributions of and can be expressed in Equations 17 and 18:

$$\pi_1^{\bullet}(\alpha \mid \underline{x}) = \Lambda^{-1}\alpha^{c+n-1} e^{-\alpha\left(n\ln\theta + d - \sum_{i=1}^{n}\ln x_i\right) - \sum_{i=1}^{n}\ln x_i} \int_1^{\infty} \beta^{-(s\alpha+1)} d\beta, \tag{17}$$

and

$$\pi_1^{\bullet}(\beta \mid \underline{x}) = \frac{1}{\beta\,\Lambda} e^{-\sum_{i=1}^{n}\ln x_i} \int_0^{\infty} \alpha^{c+n-1} e^{-\alpha\left(n\ln\theta + d + s\ln\beta - \sum_{i=1}^{n}\ln x_i\right)} d\alpha. \tag{18}$$

Here, $\Lambda = \int_1^{\infty}\int_0^{\infty} \alpha^{c+n-1} e^{-\alpha\left(n\ln\theta + d + s\ln\beta - \sum_{i=1}^{n}\ln x_i\right) - \ln\beta - \sum_{i=1}^{n}\ln x_i} \, d\alpha \, d\beta$ is the normalizing constant.

From (16) and (18), the full posterior conditional distribution, represented in Equation 19, for α given β is as follows:

$$\pi_1^{\bullet}(\alpha \mid \beta, \underline{x}) \propto \alpha^{n+c-1} e^{-\alpha\left(d + n\ln\theta + s\ln\beta - \sum_{i=1}^{n}\ln x_i\right)}. \tag{19}$$

Note that $\pi_1^{\bullet}(\alpha \mid \beta, \underline{x})$ has a gamma density with a shape parameter ($n + c$) and a scale parameter $(d + n\ln\theta + s\ln\beta - \sum_{i=1}^{n}\ln x_i)$. Therefore, the generated samples of α can be obtained by using any gamma generating routine. Similarly, from (16) and (17), the full posterior conditional distribution, represented in Equation 20, for β given α is:

$$\pi_1^{\bullet}(\beta \mid \alpha, \underline{x}) \propto \beta^{-s\alpha-1}. \tag{20}$$

Moreover, it is noticed that $\pi_1^{\bullet}(\beta \mid \alpha, \underline{x})$ has a Pareto density with a shape parameter $s\alpha$. Therefore, the generated samples of β can be easily obtained.

The Bayesian estimators of α and β under SELF, for example, $\tilde{\alpha}_{SELF}$ and $\tilde{\beta}_{SELF}$, are obtained as a posterior mean in Equations 21 and 22:

$$\tilde{\alpha}_{SELF} = E(\alpha \mid \underline{x}) = \int_0^\infty \alpha\ \pi_1^\bullet(\alpha \mid \underline{x})\ d\alpha = \Lambda^{-1}\int_1^\infty\int_0^\infty \alpha^{n+c}\beta^{-s\alpha-1} e^{-\alpha\left(d+n\ln\theta-\sum_{i=1}^n \ln x_i\right)-\sum_{i=1}^n \ln x_i}\ d\alpha\ d\beta, \quad (21)$$

$$\tilde{\beta}_{SELF} = \Lambda^{-1}\int_1^\infty\int_0^\infty \alpha^{c+n-1} e^{-\alpha\left(n\ln\theta+d+s\ln\beta-\sum_{i=1}^n \ln x_i\right)-\sum_{i=1}^n \ln x_i}\ d\alpha\ d\beta. \quad (22)$$

The Bayesian estimators of both parameters, expressed in Equations 23 and 24, under LLF, for example, $\tilde{\alpha}_{LLF}$ and $\tilde{\beta}_{LLF}$, are given by:

$$\begin{aligned}\tilde{\alpha}_{LLF} &= \left(-1/\eta\right)\ln\left(E\left[e^{-\eta\alpha}\right]\right) = \left(-1/\eta\right)\ln\left[\int_0^\infty e^{-\eta\alpha}\ \pi_1^\bullet(\alpha \mid \underline{x})d\alpha\right] \\ &= \left(-1/\eta\right)\ln\left(\frac{1}{\Lambda}\int_1^\infty\int_0^\infty \beta^{-(s\alpha+1)}\alpha^{c+n-1} e^{-\alpha\left(\eta+n\ln\theta+d-\sum_{i=1}^n \ln x_i\right)-\sum_{i=1}^n \ln x_i}\ d\alpha\ d\beta\right),\end{aligned} \quad (23)$$

$$\tilde{\beta}_{LLF} = \left(-1/\eta\right)\ln\left(\frac{1}{\Lambda}\int_1^\infty\int_0^\infty \alpha^{c+n-1} e^{-\alpha\left(n\ln\theta+d+s\ln\beta-\sum_{i=1}^n \ln x_i\right)-\sum_{i=1}^n \ln x_i-\eta\beta}\ d\alpha\ d\beta\right). \quad (24)$$

The Bayesian estimators of α and β, expressed in Equations 25 and 26, under GELF, for example, $\tilde{\alpha}_{GELF}$ and $\tilde{\beta}_{GELF}$, are given by:

$$\begin{aligned}\tilde{\alpha}_{GELF} &= E\left[\alpha^{-\varepsilon} \mid \underline{x}\right]^{-1/\varepsilon} = \left(\int_0^\infty \frac{1}{\alpha^\varepsilon}\ \pi_1^\bullet(\alpha \mid \underline{x})\ d\alpha\right)^{-1/\varepsilon} \\ &= \left(\frac{1}{\Lambda}\int_1^\infty\int_0^\infty \alpha^{c+n-(\varepsilon+1)}\beta^{-s\alpha-1} e^{-\alpha\left(n\ln\theta+d-\sum_{i=1}^n \ln x_i\right)-\sum_{i=1}^n \ln x_i}\ d\alpha\ d\beta\right)^{-1/\varepsilon},\end{aligned} \quad (25)$$

$$\tilde{\beta}_{GELF} = \left(\frac{1}{\Lambda}\int_1^\infty\int_0^\infty \alpha^{c+n-1} e^{-\alpha\left(n\ln\theta+d+s\ln\beta-\sum_{i=1}^n \ln x_i\right)-(\varepsilon+1)\ln\beta-\sum_{i=1}^n \ln x_i}\ d\alpha\ d\beta\right)^{-1/\varepsilon}. \quad (26)$$

Furthermore, the Bayesian estimators of α and β using PLF, for example, $\tilde{\alpha}_{PLF}$ and $\tilde{\beta}_{PLF}$, are obtained in Equations 27 and 28 as follows:

$$\begin{aligned}\tilde{\alpha}_{PLF} &= \left[E(\alpha^2 \mid \underline{x})\right]^{0.5} = \left[\int_0^\infty \alpha^2\ \pi_1^\bullet(\alpha \mid \underline{x})\ d\alpha\right]^{0.5} \\ &= \left(\frac{1}{\Lambda}\int_1^\infty\int_0^\infty \alpha^{c+n+1}\beta^{-s\alpha-1} e^{-\alpha\left(n\ln\theta+d-\sum_{i=1}^n \ln x_i\right)-\sum_{i=1}^n \ln x_i}\ d\alpha\ d\beta\right)^{0.5},\end{aligned} \quad (27)$$

$$\tilde{\beta}_{PLF} = \left(\frac{1}{\Lambda} \int_1^{\infty}\int_0^{\infty} \alpha^{c+n-1} e^{-\alpha\left(n\ln\theta + d + s\ln\beta - \sum_{i=1}^{n}\ln x_i\right) + \ln\beta - \sum_{i=1}^{n}\ln x_i} \, d\alpha \, d\beta \right)^{0.5}. \quad (28)$$

The Bayesian estimators of $H_\tau(X)$ and $H_q(X)$ denoted by $\tilde{H}_\tau(X)$ and $\tilde{H}_q(X)$ under SELF, LLF, GELF, and PLF are obtained by directly substituting Equations 2128 into (3) and (4), respectively. As can be seen, the mathematical forms of integrations (21)(28) are difficult to compute analytically, so the MCMC procedure is utilized to approximate these integrations. The Gibbs sampler procedure, a class of MCMC procedure to calculate the Bayes estimates, will be performed. Furthermore, the Bayesian estimators $\tilde{H}_\tau(X)$ and $\tilde{H}_q(X)$ for no-outlier (homogenous) cases for $\beta = 1$ or $s = 0$ are obtained.

SIMULATION STUDY AND RESULTS

A numerical study was conducted to assess the behavior of the ML estimate (MLE) and Bayesian estimate (BE) for the Rényi and q-entropies of PFD in the presence of outliers using different loss functions. The ML estimator of the Rényi and q-entropies was obtained. Moreover, Bayesian estimators were acquired using the uniform and gamma priors for β and α, respectively, under SELF, LLF, GELF, and PLF. Subsequently, the absolute biases (ABs) and mean squared errors (MSEs) for different sample sizes and parameter values were computed. The MCMC technique was used to generate samples from the posterior distributions. The Gibbs sampler is one of the best known MCMC sampling algorithms in the Bayesian literature. This mechanism aims to find a Markov chain that has a limiting distribution of the desired posterior, and then the simulated sample (chain) can be used to compute any required characteristic. In this study, a numerical procedure of the Gibbs sampler was implemented via R 4.0.2 program (see https://cran.r-project.org/bin/windows/base/old/4.0.2/). According to Lynch (2007), the Gibbs algorithm proceeds as follows:

Algorithm 1: The Gibbs algorithm

Step1: Initialize α_0 as the starting parameter value of α.
Step 2: For the given α_0 ,generate β_1 from the conditional distribution.
$\pi_1^*(\beta \mid \alpha, \underline{x})$.
Step 3: For the given β_1 generate α_1 from the conditional distribution.
$\pi_1^*(\alpha \mid \beta, \underline{x})$.
Step 4: Set $\alpha_0 = \alpha_1$,repeat Steps 2–3 M times, and record the sequence. (α, β) .After N burn-in iterations, the effect of the starting values is removed.

To compare the estimates, MCMC simulations were conducted for different sample sizes under SELF, LLF, GELF, and PLF. A random sample $X_1, X_2, ..., X_n$ of size n = 10, 20, 30, 40, and 50 was generated from PFD for s = 0, 1, and 2 using the quantile function. For example, at s = 1, this random sample was drawn from the PFD defined in (5) with parameters α, β ,and θ.The rest of $(n-1)$ random samples were drawn from the PFD defined in (1) with parameters α and θ. For each simulation, the parameter values were taken as $(\alpha{=}2.8,\ \beta{=}2,\ \theta{=}1.5)$, Set2 ≡ $(\alpha{=}2.5,\ \beta{=}2,\ \theta{=}1.5)$, and $\tau, q = 0.2, 0.8,$ respectively. The hyper-parameters for gamma prior were taken as $c = d = 2$. Furthermore, let η and 0.5 for LLF and for GELF. All the results were obtained based on the number of replications, i.e., M = 10,000.

The simulation results are summarized in Tables 1−4 and described through Figures 2−8. The following can be concluded:

1. The MSEs of the MLEs and BEs of the Rényi and q-entropies in the presence of outliers decreased with increasing sample sizes (Figures 2−5 and Tables 1−4).
2. The MSEs and ABs of the MLEs and BEs of the Rényi and q-entropies decreased as the exact values decreased with the number of s.
3. The MSEs and ABs of the Rényi and q-entropy estimates increased as the number of outliers increased (Tables 1−4 and Figures 4−7).
4. The MSEs and ABs of the MLEs and BEs of the Rényi and q-entropies in the presence of outliers were greater than those in the homogenous case (Figures 4−7 and Tables 1−4).
5. The MSEs and ABs of the MLEs and BEs increased with the number of outliers (Tables 1−4).
6. The MSEs of the BEs of the Rényi and q-entropies at s = 0, 1, and 2 under SELF had the smallest values as compared to the MSEs of the BEs under other loss functions (Figures 2, 3, 6, and 7 and Tables 1−4).
7. The MSEs of the BEs of the q-entropy under different loss functions at s = 0, 1, and 2 were smaller than those of the Rényi entropy.
8. History plots for different BE estimates of the Rényi and q-entropies under the four loss functions are represented in Figure 8 in the presence of outliers. The plots of chains for the Rényi and q-entropy estimates under the four loss functions looked like a horizontal band with no long upward or downward trends, which was indicative of convergence.

Figure 2

MSEs of BEs of the Q-Entropy at s = 2 for Set 1

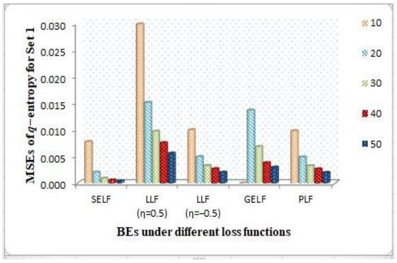


Figure 3

MSEs of BEs of the Rényi Entropy at s = 1 for Set 2

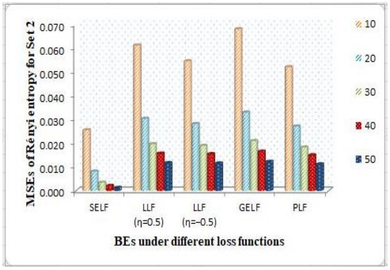


Figure 4

MSEs of $\hat{H}_{\tau}(X)$ at s = 0, 1,and 2 for Set 1

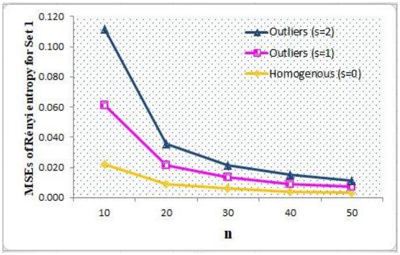


Figure 5

MSEs of $\hat{H}_{q}(X)$ at s = 0, 1, and 2 for Set 2

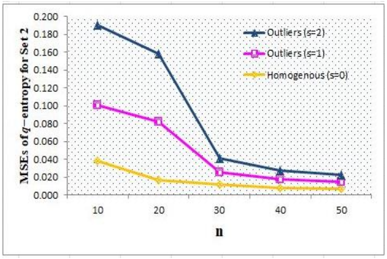


Figure 6

MSEs of BEs of Rényi Entropy at s = 0, 1, and 2 for Set 1

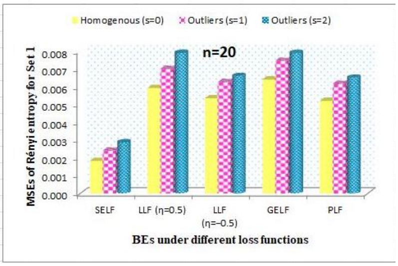


Figure 7

MSEs of BEs of Q-Entropy at s = 0, 1, and 2 for Set 2

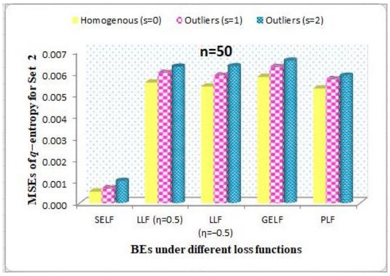
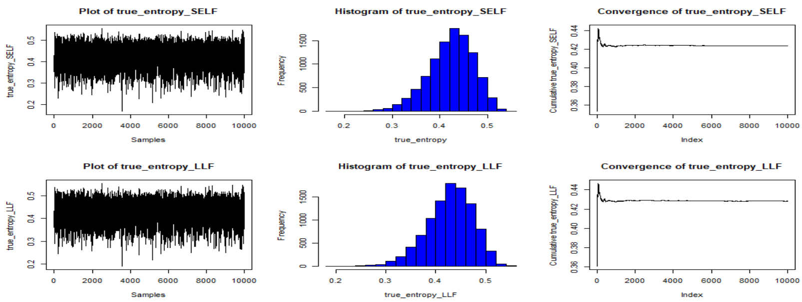


Figure 8

Bayesian Estimators of the Rényi and Q-Entropies in Presence of Outliers for Set 1 and Set 2 at Different Loss Functions

a) Rényi Entropy Estimates under SELF and LLF (η = 0.5) at n = 50 for Set 1.



b) Q-Entropy Estimates under GELF and PLF at n = 50 for Set 2

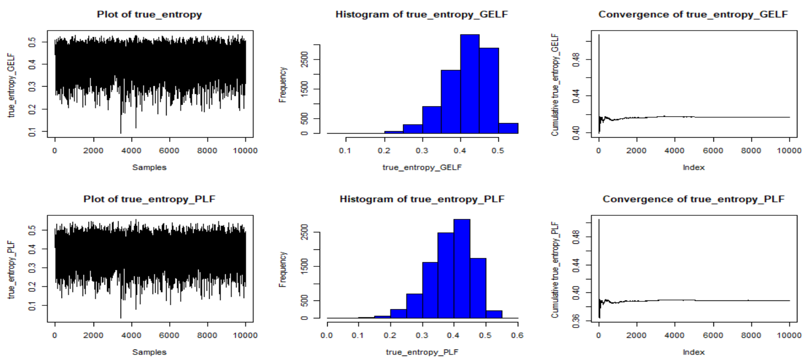


Table 1

ABs and MSEs of the Rényi Entropy Estimates for s = 0, 1and 2 at τ = 0.2

Set1 ≡ (α=2.8, β=2, θ=1.5)							
			s =0				
	n		10	20	30	40	50
	Exact value				0.40181		
	MLE	AB	0.04309	0.02221	0.01684	0.01120	0.00749
		MSE	0.02219	0.00890	0.00601	0.00411	0.00332
BE	SELF	AB	0.07045	0.04276	0.02897	0.02267	0.01813
		MSE	0.00496	0.00183	0.00084	0.00051	0.00033
	LLF (η = 0.5)	AB	0.08196	0.05136	0.03561	0.02797	0.02260
		MSE	0.01040	0.00592	0.00407	0.00333	0.00254
	LLF (η = −0.5)	AB	0.05745	0.03347	0.02192	0.01713	0.01349
		MSE	0.00885	0.00535	0.00382	0.00321	0.00245
	GELF (ε = 0.8)	AB	0.08838	0.05513	0.03820	0.02989	0.02415
		MSE	0.01156	0.00640	0.00433	0.00349	0.00264
	PLF	AB	0.06038	0.03588	0.02385	0.01867	0.01479
		MSE	0.00856	0.00520	0.00373	0.00314	0.00241
			s =1				
	Exact value		0.41335	0.40710	0.40523	0.40434	0.40382
	MLE	AB	0.08979	0.04253	0.02931	0.02024	0.01417
		MSE	0.03888	0.01260	0.00750	0.00494	0.00382
BE	SELF	AB	0.08556	0.04905	0.03264	0.02527	0.02008
		MSE	0.00732	0.00241	0.00107	0.00064	0.00040
	LLF (η = 0.5)	AB	0.09780	0.05809	0.03963	0.03082	0.02478
		MSE	0.01408	0.00699	0.00454	0.00361	0.00272
	LLF (η = −0.5)	AB	0.06970	0.03767	0.02400	0.01846	0.01418
		MSE	0.01210	0.00625	0.00424	0.00347	0.00261
	GELF (ε = 0.8)	AB	0.10355	0.06162	0.04211	0.03268	0.02630
		MSE	0.01518	0.00745	0.00479	0.00376	0.00281
	PLF	AB	0.07373	0.04097	0.02662	0.02057	0.01609
		MSE	0.01187	0.00616	0.00416	0.00341	0.00257
			s =2				
	Exact value		0.42940	0.41335	0.40907	0.40710	0.40597
	MLE	AB	0.10046	0.04467	0.03028	0.02066	0.01422
		MSE	0.05041	0.01437	0.00820	0.00529	0.00402

(continued)

			Set1 ≡ (α=2.8, β=2, θ=1.5)				
BE	SELF	AB	0.08777	0.04979	0.03359	0.02614	0.02095
		MSE	0.00785	0.00248	0.00113	0.00068	0.00044
	LLF (η = 0.5)	AB	0.09869	0.06803	0.03999	0.03125	0.02526
		MSE	0.01508	0.00791	0.00458	0.00362	0.00292
	LLF (η = −0.5)	AB	0.06970	0.04078	0.02675	0.02077	0.01647
		MSE	0.01152	0.00650	0.00424	0.00347	0.00273
	GELF (ε = 0.8)	AB	0.09824	0.06389	0.04301	0.03376	0.02748
		MSE	0.01794	0.00760	0.00484	0.00384	0.00271
	PLF	AB	0.07383	0.04355	0.02888	0.02246	0.01788
		MSE	0.01236	0.00621	0.00417	0.00342	0.00259

Table 2

ABs and MSEs of the Rényi Entropy Estimates for s = 0, 1and 2 at τ=0.8

			Set2 ≡ (α=2.5, β=2, θ=1.5)				
			s = 0				
	n		10	20	30	40	50
	Exact value				0.16516		
	MLE	AB	0.08580	0.03769	0.02905	0.01910	0.01214
		MSE	0.07980	0.03374	0.02346	0.01640	0.01352
BE	SELF	AB	0.14708	0.07686	0.05100	0.04007	0.03163
		MSE	0.02163	0.00591	0.00260	0.00161	0.00100
	LLF (η = 0.5)	AB	0.17694	0.10315	0.06910	0.05023	0.04013
		MSE	0.05648	0.02876	0.01892	0.01436	0.01086
	LLF (η = −0.5)	AB	0.10617	0.05864	0.03748	0.02955	0.02287
		MSE	0.05169	0.02340	0.01658	0.01396	0.01057
	GELF(ε = 0.8)	AB	0.19508	0.10408	0.07090	0.05547	0.04437
		MSE	0.06288	0.02771	0.01864	0.01511	0.01135
	PLF	AB	0.11529	0.06189	0.04005	0.03158	0.02459
		MSE	0.04895	0.02264	0.01614	0.01366	0.01038
			s =1				
	Exact value		0.18197	0.18122	0.18001	0.17776	0.17238
	MLE	AB	0.15162	0.07075	0.04935	0.03348	0.02265
		MSE	0.13583	0.04558	0.02844	0.01905	0.01514
	SELF	AB	0.15986	0.09016	0.05900	0.04584	0.03591
		MSE	0.02555	0.00813	0.00348	0.00210	0.00129
	LLF (η = 0.5)	AB	0.18972	0.11055	0.07424	0.05774	0.04593
		MSE	0.06117	0.03034	0.01965	0.01572	0.01172

(continued)

Set2 ≡ (α=2.5, β=2, θ=1.5)							
BE	LLF (η = −0.5)	AB	0.11895	0.06078	0.03638	0.02778	0.01963
		MSE	0.05456	0.02814	0.01896	0.01552	0.01159
	GELF(ε = 0.8)	AB	0.20786	0.12145	0.08189	0.06350	0.05065
		MSE	0.06802	0.03296	0.02099	0.01651	0.01223
	PLF	AB	0.12807	0.06906	0.04346	0.03378	0.02560
		MSE	0.05206	0.02711	0.01824	0.01501	0.01122
s =2							
	Exact value		0.18909	0.18776	0.18573	0.18238	0.17862
	MLE	AB	0.18930	0.07904	0.05314	0.03541	0.02357
		MSE	0.20040	0.05524	0.03217	0.02091	0.01622
	SELF	AB	0.18552	0.10755	0.07164	0.05566	0.04415
		MSE	0.03442	0.01157	0.00513	0.00310	0.00195
	LLF (η = 0.5)	AB	0.21026	0.12480	0.08462	0.06585	0.05267
		MSE	0.06785	0.03319	0.02103	0.01654	0.01226
BE	LLF (η = −0.5)	AB	0.15741	0.08878	0.05776	0.04491	0.03523
		MSE	0.05847	0.02970	0.01938	0.01567	0.01164
	GELF(ε = 0.8)	AB	0.22585	0.13429	0.09128	0.07085	0.05674
		MSE	0.07465	0.03585	0.02241	0.01736	0.01280
	PLF	AB	0.16227	0.09233	0.06049	0.04705	0.03700
		MSE	0.05712	0.02914	0.01904	0.01544	0.01151

Table 3

ABs and MSEs of the Q-Entropy Estimates for s = 0, 1and 2 at q = 0.2

Set1 ≡ (α=2.8, β=2, θ=1.5)							
s =0							
	n		10	20	30	40	50
	Exact value				0.31198		
	MLE	AB	0.03227	0.01716	0.01317	0.00872	0.00570
		MSE	0.01425	0.00616	0.00425	0.00295	0.00242
	SELF	AB	0.06333	0.03839	0.02603	0.02042	0.01630
		MSE	0.00401	0.00147	0.00068	0.00042	0.00027
	LLF (η = 0.5)	AB	0.14621	0.09208	0.06447	0.05070	0.01227
		MSE	0.02982	0.01510	0.00970	0.00754	0.00557
BE	LLF (η = −0.5)	AB	0.05192	0.03028	0.01991	0.01561	0.01227
		MSE	0.00697	0.00411	0.00290	0.00244	0.00185
	GELF(ε = 0.8)	AB	0.07939	0.04931	0.03414	0.02674	0.02156
		MSE	0.00931	0.00504	0.00336	0.00269	0.00203

(continued)

			Set1 ≡ (α=2.8, β=2, θ=1.5)				
	PLF	AB	0.05439	0.03234	0.02155	0.01693	0.01339
		MSE	0.00678	0.00401	0.00284	0.00239	0.00183
			s =1				
	Exact value		0.32201	0.31657	0.31495	0.31417	0.31372
	MLE	AB	0.06952	0.03402	0.02368	0.01639	0.01138
		MSE	0.02404	0.00853	0.00524	0.00351	0.00276
BE	SELF	AB	0.07780	0.04421	0.02938	0.02277	0.01805
		MSE	0.00605	0.00195	0.00086	0.00052	0.00033
	LLF (η = 0.5)	AB	0.14621	0.09208	0.06447	0.05070	0.04067
		MSE	0.02982	0.01510	0.00970	0.00754	0.00559
	LLF (η = −0.5)	AB	0.06374	0.03425	0.02186	0.01685	0.01293
		MSE	0.00980	0.00485	0.00324	0.00264	0.00198
	GELF(ε = 0.8)	AB	0.09404	0.05536	0.03771	0.02926	0.02349
		MSE	0.01252	0.00593	0.00375	0.00292	0.00217
	PLF	AB	0.06721	0.03709	0.02411	0.01867	0.01457
		MSE	0.00966	0.00481	0.00320	0.00261	0.00196
			s =2				
	Exact value		0.33606	0.32201	0.31828	0.31657	0.31558
	MLE	AB	0.07697	0.03559	0.02440	0.01670	0.01139
		MSE	0.03122	0.00979	0.00575	0.00378	0.00291
	SELF	AB	0.08582	0.04500	0.03026	0.02356	0.01883
		MSE	0.00775	0.00202	0.00092	0.00055	0.00035
	LLF (η = 0.5)	AB	0.14621	0.09208	0.06447	0.05070	0.04067
		MSE	0.02982	0.01510	0.00970	0.00754	0.00559
	LLF (η = −0.5)	AB	0.07452	0.03708	0.02429	0.01889	0.01493
		MSE	0.00999	0.00494	0.00327	0.00266	0.00200
	GELF(ε = 0.8)	AB	0.09989	0.06488	0.03799	0.02997	0.02367
		MSE	0.01367	0.00685	0.00379	0.00297	0.00217
	PLF	AB	0.07721	0.03948	0.02613	0.02034	0.01615
		MSE	0.00980	0.00489	0.00323	0.00273	0.00200

Table 4

ABs and MSEs of the Q-Entropy Estimates for s = 0, 1and 2 at q = 0.8

Set2 ≡ (α=2.5, β=2, θ=1.5)							
s = 0							
	n		10	20	30	40	50
	Exact value				0.1200		
	MLE	AB	0.05717	0.02516	0.01951	0.01278	0.00797
		MSE	0.03791	0.01652	0.01159	0.00816	0.00676
BE	SELF	AB	0.10722	0.05585	0.03710	0.02919	0.02302
		MSE	0.01150	0.00312	0.00138	0.00085	0.00053
	LLF (η = 0.5)	AB	0.12866	0.06811	0.04632	0.03643	0.02907
		MSE	0.02972	0.01311	0.00894	0.00736	0.00555
	LLF (η = −0.5)	AB	0.07807	0.04286	0.02747	0.02170	0.01679
		MSE	0.02673	0.01196	0.00843	0.00709	0.00537
	GELF(ε = 0.8)	AB	0.14188	0.07539	0.05134	0.04019	0.03212
		MSE	0.03315	0.01436	0.00959	0.00775	0.00581
	PLF	AB	0.08443	0.04513	0.02928	0.02313	0.01800
		MSE	0.02544	0.01160	0.00822	0.00695	0.00528
s =1							
	Exact value		0.12774	0.12720	0.12635	0.12475	0.12092
	MLE	AB	0.10116	0.04805	0.03368	0.02286	0.01536
		MSE	0.06246	0.06547	0.01392	0.00941	0.00753
BE	SELF	AB	0.11630	0.00429	0.04286	0.03334	0.02609
		MSE	0.01353	0.00441	0.00184	0.00111	0.00068
	LLF (η = 0.5)	AB	0.13774	0.08005	0.05374	0.04182	0.03323
		MSE	0.03214	0.01572	0.01011	0.00806	0.00599
	LLF (η = −0.5)	AB	0.08715	0.04457	0.02679	0.02051	0.01453
		MSE	0.02823	0.01433	0.00962	0.00787	0.00587
	GELF(ε = 0.8)	AB	0.15096	0.08792	0.05923	0.04595	0.03661
		MSE	0.03581	0.01711	0.01081	0.00848	0.00626
	PLF	AB	0.09351	0.05041	0.03178	0.02475	0.01875
		MSE	0.02705	0.01387	0.00929	0.00763	0.00570
s =2							
	Exact value		0.12825	0.12812	0.12730	0.12575	0.12569
	MLE	AB	0.12483	0.07344	0.03618	0.02413	0.01595
		MSE	0.08996	0.07647	0.01569	0.01031	0.00806
	SELF	AB	0.13474	0.07790	0.06187	0.04033	0.03196
		MSE	0.01816	0.00607	0.00569	0.00163	0.00102

(continued)

Set2 ≡ (α=2.5, β=2, θ=1.5)							
	LLF (η = 0.5)	AB	0.15257	0.09025	0.06115	0.04760	0.03803
		MSE	0.03571	0.01722	0.01083	0.00849	0.00628
	LLF (η = −0.5)	AB	0.11460	0.06449	0.04198	0.03268	0.02561
		MSE	0.03053	0.01527	0.00990	0.00799	0.00593
BE	GELF(ε = 0.8)	AB	0.16398	0.09712	0.06594	0.05120	0.04095
		MSE	0.03935	0.01863	0.01156	0.00893	0.00656
	PLF	AB	0.11798	0.06699	0.04391	0.03419	0.02686
		MSE	0.02988	0.01500	0.00974	0.00788	0.00586

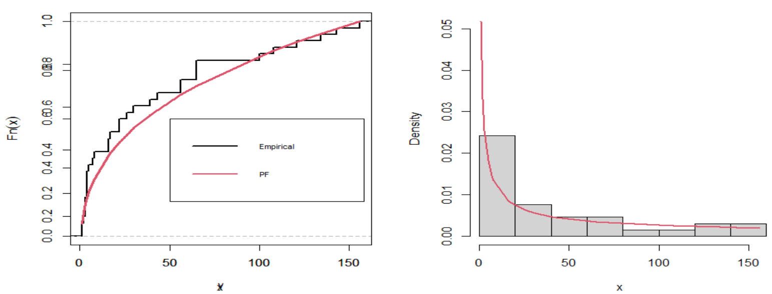
ILLUSTRATIVE EXAMPLE

The real datasets were utilized to verify the proposed estimators examined in the simulation study.

Dataset 1: In an early paper on regression analysis of lifetime data, Feigl and Zelen (1965) provided data on survival times for 33 patients suffering from acute myelogenous leukemia. These survival times depended on several factors, such as age, time of diagnosis, and the body's response to treatment. In this regard, survival times between patients were different, which led to the same distribution with different parameters. The present study used the Kolmogorov–Smirnov (KS) test for the real dataset, and its p-value implied that PFD in the presence of outliers fitted the data. The estimated PDF and CDF for leukemia data are demonstrated in Figure 9.

Figure 9

Plots of the Estimated PDF and CDF for Leukemia Data



Therefore, on the basis of the real data, the estimates of the entropies were computed using the proposed estimation method. The results are listed in Table 5.

Table 5

Estimates of the Entropies and their MSEs (in Brackets) for Leukemia Data

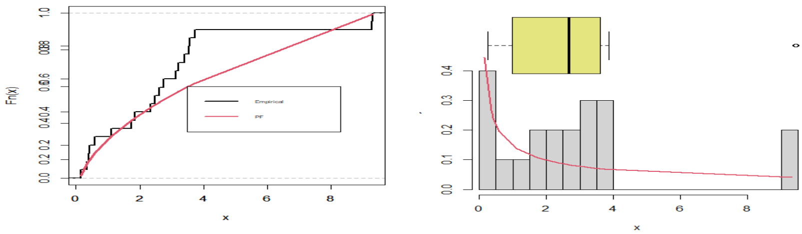
		Rényi entropy			q−entropy		
	MLE	s = 0	s = 1	s = 2	s = 0	s = 1	s = 2
		0.14726 (0.06169)	0.16520 (0.07787)	0.26470 (0.08089)	0.92345 (0.34769)	0.94560 (0.38630)	0.97865 (0.47887)
BE	SELF	0.33642 (0.08225)	0.29307 (0.07924)	0.38051 (0.08452)	0.70851 (0.04964)	0.71536 (0.06920)	0.71372 (0.08369)
	LLF ($\eta = 0.5$)	0.34580 (0.08280)	0.40411 (0.08542)	0.48392 (0.09466)	0.714621 (0.05239)	0.713202 (0.06727)	0.817984 (0.08326)
	LLF ($\eta = -0.5$)	0.34133 (0.08254)	0.29695 (0.07954)	0.31366 (0.08077)	0.71184 (0.05113)	0.71536 (0.06846)	0.81501 (0.08443)
	GELF ($\varepsilon = 0.8$)	0.29610 (0.07947)	0.26368 (0.07670)	0.34233 (0.08260)	0.71406 (0.05214)	0.71555 (0.06861)	0.71566 (0.08481)
	PLF	0.35792 (0.08345)	0.30928 (0.08046)	0.40217 (0.08536)	0.70682 (0.04889)	0.71532 (0.06969)	0.71289 (0.08321)

The observed results showed that the MSEs of the MLEs and BEs of $H_\tau(X)$ and $H_q(X)$in the presence of outliers were larger than those in the homogenous case. The MSEs of the BEs of $H_q(X)$ under different loss functions at s = 0, 1, and 2 were smaller than those of $H_\tau(X)$. In addition, it can be concluded that the entropy estimates increased with s; i.e., the estimated values of the entropies increased as the number of outliers increased.

Dataset 2: The real dataset was studied by Dixit and Nooghabi (2011). The data represented the lifetime distribution of 20 electronic tubes with insufficient power supply. It was observed that some tubes (1–2) were of different quality. The p-value of the KS test showed that the PFD fitted this real dataset. The estimated PDF and CDF are presented in Figure 10.

Figure 10

Plots of the Estimated PDF and CDF for 20 Electronic Tubes



Therefore, on the basis of the real data, the estimates of the entropies were computed using the estimation method proposed herein. The results are presented in Table 6.

Table 6

Estimates of the Entropies and their MSEs (in Brackets) for 20 Electronic Tubes

		Rényi entropy			q–entropy		
		s = 0	s = 1	s = 2	s = 0	s = 1	s = 2
	MLE	0.29739 (0.18124)	0.39739 (0.19691)	0.42650 (0.26050)	0.85267 (0.04558)	0.86243 (0.05370)	0.93729 (0.10449)
BE	SELF	0.41297 (0.08101)	0.42389 (0.09040)	0.42806 (0.09522)	0.77034 (0.01474)	0.83674 (0.04245)	0.83822 (0.04674)
	LLF (η = 0.5)	0.41119 (0.08000)	0.41786 (0.08682)	0.42514 (0.09342)	0.79960 (0.02270)	0.81842 (0.02813)	0.86441 (0.06269)
	LLF (η = −0.5)	0.41336 (0.08001)	0.41801 (0.08689)	0.41973 (0.09015)	0.71971 (0.00501)	0.82683 (0.03847)	0.83619 (0.04087)
	GELF (ε = 0.8)	0.40744 (0.07789)	0.41265 (0.08377)	0.41721 (0.08864)	0.82012 (0.02931)	0.85247 (0.04918)	0.85289 (0.05705)
	PLF	0.41603 (0.08276)	0.43149 (0.09503)	0.43215 (0.09776)	0.73575 (0.00754)	0.82784 (0.03887)	0.81823 (0.04169)

The observed results showed that the MSEs of the MLEs and BEs of $H_\tau(X)$ and $H_q(X)$ at $s = 1$ and $s = 2$ were larger than those at $s = 0$. The MSEs of the BEs of $H_q(X)$ under different loss functions at $s = 0$, 1, and 2 were smaller than those of $H_\tau(X)$. Finally, it can be concluded that the entropy estimates increased with s; i.e., the estimated values of the entropies increased as the number of outliers increased.

CONCLUSION

This paper proposes an estimation method of $H_{t}(X)$ and $H_{q}(X)$ for PFD in the presence of no outliers and *s* outliers. The ML estimators of the Rényi and *q*-entropies were obtained. The Bayesian estimators under uniform and gamma priors were derived for several loss functions. This study employed the MCMC procedure to obtain the Bayes estimates based on Gibbs sampling. The performance of the entropy estimates for PFD was examined in terms of their ABs and MSEs based on 10,000 replications. Real data analysis and simulation studies were conducted. The numerical results of the simulation study indicated that the MSEs of the ML and BEs of the entropies decrease with the sample sizes. The MSEs of both entropy estimates in the no outlier case are better than those in the outlier case. Generally, the MSEs of the Bayesian entropies under SELF are smaller than the MSEs of other loss functions in a majority of the investigated cases.

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