JOURNAL OF INFORMATION AND COMMUNICATION TECHNOLOGY
http://e-journal.uum.edu.my/index.php/jict

How to cite this article:
Ibrahim, H., Aldiabat, R., & Karim, S. (2022). Butterfly triple system algorithm based on graph theory. *Journal of Information and Communication Technology, 21*(1), 27-49. https://doi.org/10.32890/jict2022.21.1.2

Butterfly Triple System Algorithm Based on Graph Theory

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Received: 8/11/2020 Revised: 22/6/2021 Accepted: 5/7/2021 Published: 11/11/2021

ABSTRACT

In combinatorial design theory, clustering elements into a set of three elements is the heart of classifying data, which has recently received considerable attention in the fields of network algorithms, cryptography, design and analysis of algorithms, statistics, and information theory. This article provides insight into formulating an algorithm for a new type of triple system, called a Butterfly triple system. Basically, in this algorithm development, a starter of cyclic near-resolvable $\left(\frac{v-1}{2}\right)$-cycle system of the 2-fold complete graph $2K_v$ was employed to construct the starter of cyclic $\left(\frac{v-1}{2}\right)$- star decomposition of $2K_v$. These starters were then decomposed into triples and classified as a starter of a cyclic Butterfly triple system. The obtained starter set generated a triple system of order v. A special reference for case $v \equiv 9 \pmod{12}$ was presented to demonstrate the development of the Butterfly triple system.

Keywords: Cyclic triple system, graph decompositions, λ-fold complete graph.

INTRODUCTION

In this paper, all graphs are considered to be undirected odd order vertices in $\mathbb{Z}_v$. K_v will denote the complete graph of order v, λK_v and will denote the λ-fold complete graph of order v, which is obtained by substituting each edge in K_v by λ parallel edges. A k-cycle, written $C_k = (c_1, c_2, \ldots, c_k)$, consists of k distinct vertices $\{c_1, c_2, \ldots, c_k\}$ and k edges $\{c_i, c_{i+1}\}$, $1 \leq i \leq k$, where $c_{k+1} = c_1$. A k-star, written $S_k = (v_0;\ v_1, v_2, \ldots, v_k)$, is a graph with one vertex v_0 of degree k and k vertices $\{v_1, v_2, \ldots, v_k\}$ of degree one.

Let H and G be graphs. An H-decomposition of G (or a (G, H)-design) is a collection $\mathcal{H}$ of subgraphs of G, each isomorphic to H, whose edges partition the edges of G (Heinrich, 1996). When k-is a H cycle, such a decomposition is known as a k-cycle system of G. A (G, H)-design with vertices in $\mathbb{Z}_v$ is said to be cyclic if $\mathcal{H} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ is the collection of all subgraphs in $(\lambda K_v, H)$- design. Then, there is also $\mathcal{H} = \{\Gamma_1 + 1, \Gamma_2 + 1, \ldots, \Gamma_n + 1\}$, and it is said to be simple if $\mathcal{H}$ contains no repeated subgraphs.

Let Γ be a member of cyclic $(\lambda K_v, H)$-design. Then, an orbit of Γ, denoted by $orb(\Gamma)$, is defined by the set $\{\Gamma + i \pmod v \mid i \in \mathbb{Z}_v\}$. The orbit of Γ is called full if its cardinality is v; otherwise it is considered short. Any cyclic $(\lambda K_v, H)$-design should be generated by the orbit of graphs (called a starter of cyclic $(\lambda K_v, H)$-design) (Wu & Lu, 2008).

An m-factor of λK_v is a spanning subgraph of λK_v in which every vertex has the degree m. A near-m-factor of λK_v is an m-factor of $\lambda K_v - a$ for some vertex a in λK_v. In a k-cycle system of $2K_v$, if the collection of cycles can be partitioned into near-2-factors $\mathcal{N}_0, \mathcal{N}_1, \ldots, \mathcal{N}_{v-1}$, then the k-cycle system of $2K_v$ is said to be near-resolvable and denoted by $(v, k, 2)$-$NRCS$. Obviously, the existence of near-resolvable k-cycle system of $2K_v$ implies that λ must be even and k divides $v - 1$ (Rodger, 1996; Ferber & Kwan, 2020). Recently, the existence of a cyclic $(v, k, 2)$-$NRCS$ has been proved for $k = \frac{v-1}{2}$ with $v \equiv 9 \pmod{12}$ by Aldiabat et al. (2019) and for $k = 4$ with $v \equiv 1 \pmod 4$ by Matsubara and Kageyama (2019).

A balanced incomplete block design $(BIBD)$ is a kind of combinatorial clustering algorithm. It is defined as a collection of k-subsets (called blocks) of a v-set V, $2 \leq k < v$, such that each pair of distinct points of

V is contained in exactly λ blocks. The design is often written as a (v,k,λ)-$BIBD$ or a (v,k,λ) -design. A (v,k,λ)-$BIBD$ is said to be cyclic if $V=\mathbb{Z}_v$ and if it can be generated from a subcollection of its blocks (called the starter of cyclic (v,k,λ)-$BIBD$) by repeatedly adding 1 modulo . For more recent developments in clustering algorithms, see Hairuddin et al. (2020), Seman and Sapawi (2018), and Swesi and Bakar (2019). A λ-fold triple system of order v (or a triple system of order v and index λ), denoted by $TS(v,\lambda)$, is a $(v,3,\lambda)$-$BIBD$ and the blocks are called triples. It can be depicted as a decomposition of λK_v into triangles.

The study of λ -fold triple systems is an interesting area in combinatorial design theory due to its applicability in a wide range of areas, such as tournament scheduling, computational biology, communications engineering, design and analysis of algorithms, network design, information theory, cryptography, coding theory, and optical orthogonal codes (Chen & Wei, 2012; Kaski, Östergard, & Patric, 2006). The necessary and sufficient conditions for the existence of $TS(v,\lambda)$ have been established by Hanani (1961). The same results have been proven (in an easier way) by Nash-Williams (1972) and Hwang and Lin (1974). The existence of cyclic $TS(v,\lambda)$ for $v\equiv 1,3 \pmod 6$ has been verified by Colbourn and Colbourn (1981). Colbourn and Rosa (1999) gave the values of and for which a cyclic $TS(v,\lambda)$ exists. The problem of constructing triple systems and related areas remains very active in recent years (Ferber & Kwan, 2020; Ballico, Favacchio, Guardo, & Milazzo, 2021; Daniel & Bridget, 2021). The following section highlights some of the exciting recent developments of triple systems that have motivated the present authors to undertake this study.

A triad design is one of the contemporary triple systems, which are concerned with arranging a set of unordered triples on v objects into v classes satisfying certain specified conditions. This design originally arose from a request to construct tournaments appropriate for use in a paintball game in which three teams compete at a time. The existence of a triad design for $v\equiv 1$ or $5 \pmod 6$ was formulated and proved (Ibrahim, 2006). Then, a new 3-fold triple system called compatible factorization to complete the triad design was developed for the compatible factorization for every odd order $v>3$ (Ibrahim, 2006). New algorithms for a cyclic triad design for the cases $v\equiv 1$ or $5 \pmod 6$ and some related results to the triad design were constructed (Ibrahim, Abu Saa, & Kalmoun, 2011).

In 2010, Tian and Wei constructed some decompositions of $\mathbb{Z}_v$ the set of all triples of into small cyclic triple systems for an v odd using the method of constructing the difference triples. In like manner, a decomposition of all triples of $\mathbb{Z}_v$ into simple cyclic triple systems for being a prime power or thrice a prime power was constructed by Tian and Wei (2013). Recently, motivated by constructing cyclic triple systems for arranging all triples of $\mathbb{Z}_v$ into v rows with certain constraints and repetition of triples for $v \equiv 3 \pmod 6$, a new type of cyclic 3-fold triple system called near-compatible factorization was constructed (Aldiabat, Ibrahim & Karim, 2019).

This paper produces a new type of cyclic triple systems for arranging $v(v-1)$ triples into v rows, called a Butterfly triple system (BTS), using a construction of simple cyclic $\left(v, \frac{v-1}{2}, 2\right)$-$NRCS$ focus on case $v \equiv 9 \pmod{12}$.

PRELIMINARIES

This section provides some definitions, notations, and results that are needed in the formulation of the main results.

Definition 1 (Buratti, Capparelli, & Del Fra, 2010). Let Γ be a subgraph of λK_v, the stabilizer of under the action of $\mathbb{Z}_v$ is defined by $stab(\Gamma) = \{z \in \mathbb{Z}_v \mid z + \Gamma = \Gamma\}$. The stabilizer of Γ is called trivial if $stab(\Gamma) = \{0\}$.

Definition 2 (Costa et al., 2018). LetLet $\Gamma = (V(\Gamma), E(\Gamma))$ be a subgraph of λK_v. The list of differences from Γ in λK_v is:

$$D(\Gamma) = \{min\{|y-x|, v-|y-x|\} \mid x, y \in V(\Gamma), \{x, y\} \in E(\Gamma)\}.$$

The next lemma is a particular consequence of the results developed by Buratti (2000). It is crucial in proving the existence of cyclic (G, H)-design and in constructing its starter.

Lemma 3. Let δ be a multiset of subgraphs of λK_v, whereby each has a trivial stabilizer. Then, δ is a starter of cyclic $(\lambda K_v, H)$-design if and only if $D(\delta)$ covers each nonzero integer $\mathbb{Z}_{\frac{v+1}{2}}$ of exactly λ times.

The following paragraphs review the definitions of relative path, relative cycle, and alternating arithmetic path that will be the basis for constructing a starter of simple cyclic $(12n+3, 6n+1, 2)$-$NRCS$.

Definition 4 (Aldiabat et al., 2019). Let G be a graph of order v, $P_k = [x_1, x_2, \dots, x_k]$ be a k-path of G, and $C_k = (x_1, x_2, \dots, x_k)$ be a k-cycle of G.

(i) The k-path $\overline{P}_k = [v - x_1, v - x_2, \dots, v - x_k]$ is called the relative path of P_k.

(ii) The k-cycle $\overline{C}_k = (v - x_1, v - x_2, \dots, v - x_k)$ is called the relative cycle of C_k.

Definition 5 (Aldiabat et al., 2019). Let m and n be positive integers with $n \le m \le n+1$. An $m+n$-alternating arithmetic path, denoted by $AAP(m+n)$, is a path of length $m+n$ with vertex set $V = \{x_1, x_2, \dots, x_m\} \cup \{y_1, y_2, \dots, y_n\}$ edge set $E = \{\{x_i, y_i\} | \ 1 \le \ \le n\} \cup \{\{y_i, x_{i+1}\} | \ 1 \le i \le m-1\}$, such that the following properties are satisfied:

(i) $x_{i+1} - x_i$ is constant for all $1 \le i \le m-1$.

(ii) $y_{i+1} - y_i$ is constant for all $1 \le i \le n-1$.

According to Definition 5, the $(m+n)$-alternating arithmetic path either has an odd order $(2n+1)$ when $m = n+1$, or has an even order $(2n)$ when $m = n$. This study used the following notations for $(m+n)$-alternating arithmetic path of odd order and even order, respectively:

$$AAP(2n+1) = [f(1), g(1), f(2), g(2), \dots, f(n), g(n), f(n+1)] = [f(i), g(i)]_{2n+1},$$

$$AAP(2n) = [f(1), g(1), f(2), g(2), \dots, f(n), g(n)] = [f(i), g(i)]_{2n}.$$

Example 6 demonstrates the above concepts.

Example 6. Let $G = 2K_9$ and $P_6 = [0, 3, 1, 5, 2, 7]$ be a 6-path of G.
Based on Definition 5, P_6 is a 6-alternating arithmetic path that can be written as: $AAP(6) = [0, 3, 1, 5, 2, 7] = [i - 1, 2i + 1]_6$.

Theorem 7 (Aldiabat et al., 2019). There exists a simple cyclic $\left(v, \frac{v-1}{2}, 2\right)$-$NRCS$ for $v \equiv 9 \pmod{12}$.

Definition 8 (Abel & Buratti, 2006). Let B be a k-subset of $\mathbb{Z}_v$. The list of difference from B is the multiset:

$$D(B) = \{min\{|b-a|, v - |b-a|\} \mid \ a \neq b \in B\}.$$

Lemma 9 (Abel & Buratti, 2006). Let $\mathcal{B}$ be a multiset of k-subsets of $\mathbb{Z}_v$. Then, $\mathcal{B}$ is a starter of cyclic (v, k, λ)-$BIBD$ if and only if each nonzero integer of $\mathbb{Z}_{\frac{v+1}{2}}$ occurs λ times in $D(\mathcal{B})$.

CYCLIC NEAR-RESOLVABLE $(6n + 4)$-CYCLE SYSTEM OF $2K_{12n+9}$ AND CYCLIC $(6n + 4)$-STAR DECOMPOSITION OF $2K_{12n+9}$

In this section, a construction for a starter of simple cyclic $(12n + 9, 6n + 4, 2)$-$NRCS$ is presented. Then, this construction is used to construct a starter of cyclic $(6n + 4)$-star decomposition of $2K_{12n+9}$. These types of constructions are the basis for constructing a cyclic triple system in the next section.

Algorithm for a Simple Cyclic Near-Resolvable $(6n + 4)$-Cycle System of $2K_{12n+9}$

Let $n \geq 0$, $C_{(6n+4)_1}$ and $C_{(6n+4)_2}$ be the $(6n + 4)$-cycles of $2K_{12n+9}$ defined as linked vertex-disjoint paths as follows:

$$C_{(6n+4)_1} = \big(AAP_1(4n + 2), AAP_2(n + 1), AAP_3(n + 1)\big),$$
$$C_{(6n+4)_2} = \Big(\overline{AAP_1}(4n + 2), \overline{AAP_2}(n + 1), \overline{AAP_3}(n + 1)\Big), \quad (1)$$

where

$$AAP_1(4n + 2) = [4i - 2, 12n - 4i + 9]_{4n+2},$$
$$AAP_2(n + 1) = [4i - 3, 12n - 4i + 10]_{n+1},$$
$$AAP_3(n + 1) = \begin{cases} [10n - 4i + 10, 2n + 4i + 1]_{n+1} & \text{if } n \text{ is odd,} \\ [2n + 4i - 1, 10n - 4i + 8]_{n+1} & \text{if } n \text{ is even.} \end{cases}$$
$$\overline{AAP_1}(4n + 2) = [v - (4i - 2), v - (12n - 4i + 9)]_{4n+2} = [12n - 4i + 11, 4i]_{4n+2},$$
$$\overline{AAP_2}(n + 1) = [v - (4i - 3), v - (12n - 4i + 10)]_{n+1} = [12n - 4i + 12, 4i - 1]_{n+1},$$
$$\overline{AAP_3}(n + 1) = \begin{cases} [v - (10n - 4i + 10), v - (2n + 4i + 1)]_{n+1} & \text{if } n \text{ is odd,} \\ [v - (2n + 4i - 1), v - (10n - 4i + 8)]_{n+1} & \text{if } n \text{ is even.} \end{cases}$$
$$= \begin{cases} [2n + 4i - 1, 10n - 4i + 8]_{n+1} & \text{if } n \text{ is odd,} \\ [10n - 4i + 10, 2n + 4i + 1]_{n+1} & \text{if } n \text{ is even.} \end{cases}$$

For $n \geq 0$, any $(4n + 2)$-alternating arithmetic path has an even order. In addition, any $(n + 1)$-alternating arithmetic path has an even order when $n \geq 0$ is odd, while any $(n + 1)$-alternating arithmetic path has an odd order when $n \geq 0$ is even. Figures 1 and 2 illustrate the constructions of $C_{(6n+4)_1}$ and $C_{(6n+4)_2}$ in terms of their vertices, where $C_{(6n+4)_i} =$

$(c_{i,1}, c_{i,2}, \dots, c_{i,6n+1})$for $i = 1, 2$, and n is odd and even, respectively. As shown in Figures 1 and 2, the construction for a starter of simple cyclic $(12n + 9, 6n + 4, 2)$-$NRCS$ has a butterfly shape in which every cycle represents a side of symmetrical butterfly wings. If one cycle of the starter is given, then the other is the relative cycle. Furthermore, the construction is described as linked alternating arithmetic paths in which the vertices are distinguished by two different colors that show the pattern. The next example is a construction of simple cyclic $(9, 4, 2)$-$NRCS$ in accordance with the construction in Equation 1.

Example 10. The starter of simple cyclic $(9, 4, 2)$-$NRCS$ is given by the set $\mathcal{S} = \{C_{4_1}, C_{4_2}\}$where $C_{4_1} = (2, 5, 1, 3)$and $C_{4_2} = (7, 4, 8, 6)$,which is obtained by substituting $n = 0$ into Equation 1.

It can be easily checked that C_{4_2} is the relative cycle of C_{4_1}. Now, all the cycles of simple cyclic $(9, 4, 2)$-$NRCS$ can be generated from $\mathcal{S}$ by repeatedly adding 1 modulo v, as shown in Table 1.

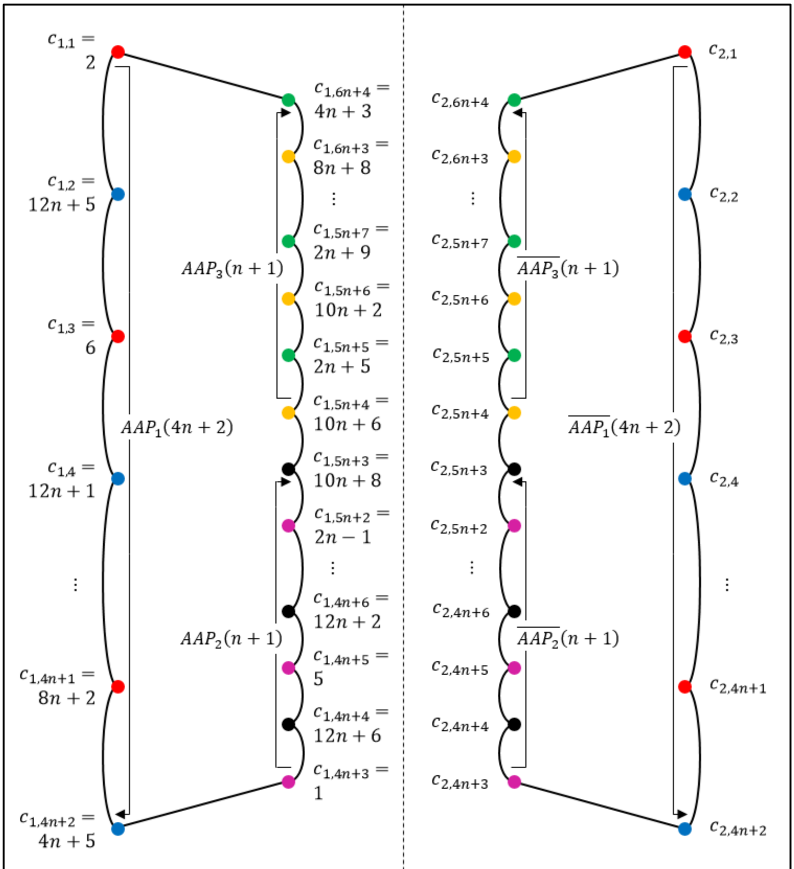
Table 1

A Simple Cyclic $(9, 4, 2)$-$NRCS$

Focus	$Orb(C_{4_1})$	$Orb(C_{4_2})$
$i = 0$	$(2, 5, 1, 3)$	$(7, 4, 8, 6)$
$i = 1$	$(3, 6, 2, 4)$	$(8, 5, 0, 7)$
$i = 2$	$(4, 7, 3, 5)$	$(0, 6, 1, 8)$
⋮	⋮	⋮
$i = 8$	$(1, 4, 0, 2)$	$(6, 3, 7, 5)$

Figure 1

The Construction of $C_{(6n+4)_1}$ and $C_{(6n+4)_2}$ in $2K_{12n+9}$, for odd $n \geq 0$



$AAP_1(4n + 2) = [4i - 2, 12n - 4i + 9]_{4n+2}$

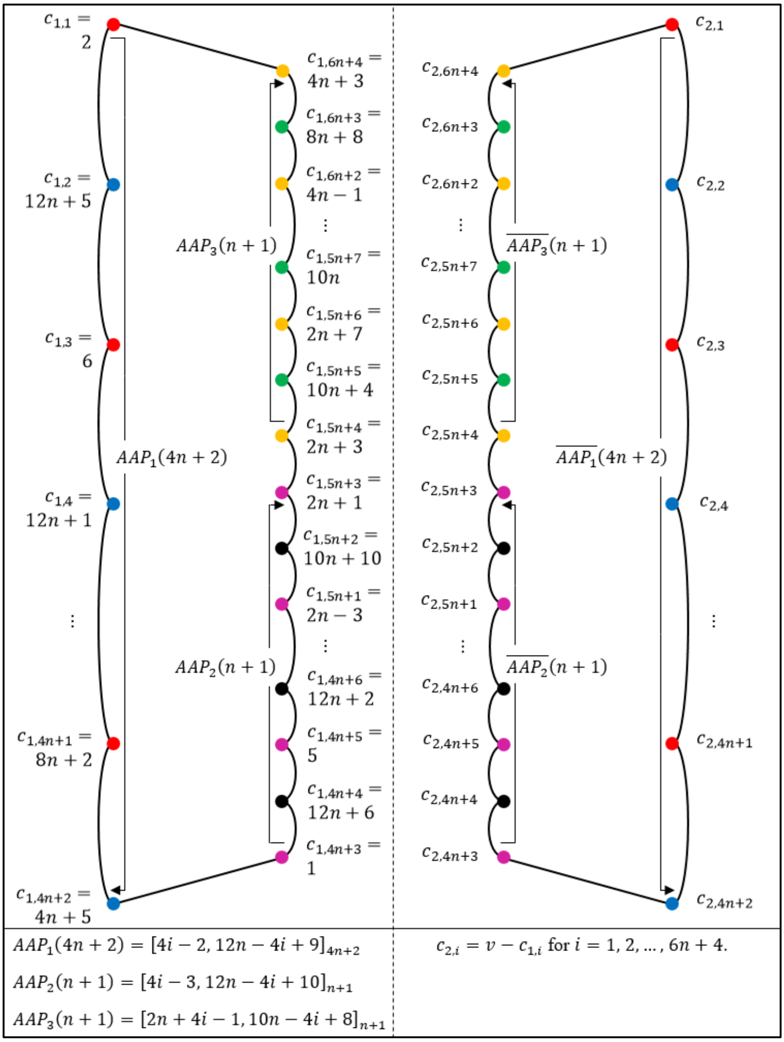
$AAP_2(n + 1) = [4i - 3, 12n - 4i + 10]_{n+1}$

$AAP_3(n + 1) = [10n - 4i + 10, 2n + 4i + 1]_{n+1}$

$c_{2,i} = v - c_{1,i}$ for $i = 1, 2, \ldots, 6n + 4$.

Figure 2

The Construction of $C_{(6n+4)_1}$ *and* $C_{(6n+4)_2}$ *in* $2K_{12n+9}$*, for even* $n \geq 0$



Algorithm for a Simple Cyclic $(6n + 4)$- Star Decomposition of $2K_{12n+9}$

Consider the construction of $C_{(6n+4)_i}$, for $i = 1, 2,$ defined in Equation 1. Let $S_{(6n+4)_i}$, for $i = 1, 2,$ be the $(6n + 4)$- star of $2K_{12n+9}$ defined by:

$$S_{(6n+4)_i} = (0;\ c_{i,1}, c_{i,2}, \dots, c_{i,6n+4}), \qquad (2)$$

where $c_{i,j} \in V(C_{(6n+4)_i})$ for $1 \leq j \leq 6n + 4$.

Theorem 11. For any $n \geq 0$, there exists a cyclic $(6n + 4)$-star decomposition of $2K_{12n+9}$.

Proof. To prove that the set $\mathcal{F} = \{S_{(6n+4)_1}, S_{(6n+4)_2}\}$ defined in Equation 2 is a starter for cyclic $(6n + 4)$-star decomposition of $2K_{12n+9}$, there is a need to find the list of differences from $\mathcal{F}$ and the stabilizer of each $(6n + 4)$-star in $\mathcal{F}$ (according to Lemma 9). Based on Definition 2, the list of differences from $\mathcal{F}$ is:

$$D(F) = D(S_{(6n+4)_1}) \uplus D(S_{(6n+4)_2}), \quad (3)$$

where

$$D(S_{(6n+4)_1}) = \{min\{|c_{1,j} - 0|, v - |c_{1,j} - 0|\} \mid 1 \leq j \leq 6n + 4\} = \{1, 2, \dots, 6n + 4\},$$
$$D(S_{(6n+4)_2}) = \{min\{|c_{2,j} - 0|, v - |c_{2,j} - 0|\} \mid 1 \leq j \leq 6n + 4\} = \{1, 2, \dots, 6n + 4\}.$$

From Equation (3), $D(F)$ covers each nonzero element of $\mathbb{Z}_{6n+5}$ exactly twice.
Based on Definition 1, for $i = 1, 2$, the stabilizer of $S_{(6n+4)_i}$ is defined by:
$stab(S_{(6n+4)_i}) = \{z \in \mathbb{Z}_v \mid z + S_{(6n+4)_i} = S_{(6n+4)_i}\}$.
Suppose $z \in stab(S_{(6n+4)_i})$, then:

$$z + S_{(6n+4)_i} = S_{(6n+4)_i},$$
$$(z;\ c_{i,1} + z, c_{i,2} + z, \dots, c_{i,6n+4} + z) = (0;\ c_{i,1}, c_{i,2}, \dots, c_{i,6n+4}).$$

Thus, $z = 0$, and so, for $i = 1, 2$, $S_{(6n+4)_i}$ has a trivial stabilizer.
Therefore, from Lemma 3, $\mathcal{F} = \{S_{(6n+4)_1}, S_{(6n+4)_2}\}$ is a starter for cyclic $(6n + 4)$-star decomposition of $2K_{12n+9}$. ϒ

CYCLIC BUTTERFLY TRIPLE SYSTEM

In this section, the Butterfly triple system is defined. Then, the existence of a cyclic Butterfly triple system for $v \equiv 9 \pmod{12}$ is verified Finally, a construction method is developed to construct such triple system for $v \equiv 9 \pmod{12}$ using the constructions presented in the previous section.

Definition 12. A Butterfly triple system on v objects, denoted by $BTS(v)$, is a $v \times (v - 1)$ array of triples that satisfy the following conditions:
(i) Object i is contained in each triple of row i.
(ii) Each object except i is contained in exactly two triples of row i.

From the above definition, it is easy to see that row i forms a near-two-factor with focus i by removing object i from each triple in row i. Therefore, to produce a $BTS(9)$, nine rows with eight columns are needed as illustrated in Table 2.

Table 2

An Example of $BTS(9)$

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
R_0	{0, 2, 5}	{0, 5, 1}	{0, 1, 3}	{0, 3, 2}	{0, 7, 4}	{0, 4, 8}	{0, 8, 6}	{0, 6, 7}
R_1	{1, 3, 6}	{1, 6, 2}	{1, 2, 4}	{1, 4, 3}	{1, 8, 5}	{1, 5, 0}	{1, 0, 7}	{1, 7, 8}
R_2	{2, 4, 7}	{2, 7, 3}	{2, 3, 5}	{2, 5, 4}	{2, 0, 6}	{2, 6, 1}	{2, 1, 8}	{2, 8, 0}
R_3	{3, 5, 8}	{3, 8, 4}	{3, 4, 6}	{3, 6, 5}	{3, 1, 7}	{3, 7, 2}	{3, 2, 0}	{3, 0, 1}
R_4	{4, 6, 0}	{4, 0, 5}	{4, 5, 7}	{4, 7, 6}	{4, 2, 8}	{4, 8, 3}	{4, 3, 1}	{4, 1, 2}
R_5	{5, 7, 1}	{5, 1, 6}	{5, 6, 8}	{5, 8, 7}	{5, 3, 0}	{5, 0, 4}	{5, 4, 2}	{5, 2, 3}
R_6	{6, 8, 2}	{6, 2, 7}	{6, 7, 0}	{6, 0, 8}	{6, 4, 1}	{6, 1, 5}	{6, 5, 3}	{6, 3, 4}
R_7	{7, 0, 3}	{7, 3, 8}	{7, 8, 1}	{7, 1, 0}	{7, 5, 2}	{7, 2, 6}	{7, 6, 4}	{7, 4, 5}
R_8	{8, 1, 4}	{8, 4, 0}	{8, 0, 2}	{8, 2, 1}	{8, 6, 3}	{8, 3, 7}	{8, 7, 5}	{8, 5, 6}

As shown in row R_0, the object 0 is contained in each triple and every object other than 0 is contained in exactly two triples. Then, by isolating the object 0 from each triple of R_0, R_0 is obtained that satisfies a near-two-factor with focus zero.

For example, the corresponding near-two-factor to row R_0 is:

F_0: 0, $\{\{1,6\},\{6,5\},\{5,7\},\{7,1\},\{8,3\},\{3,4\},\{4,2\},\{2,8\}\}$.

Therefore, row R_i satisfies a near-two-factor with focus i.

The Existence of a Cyclic Butterfly Triple System

In this short subsection, the existence of a cyclic Butterfly triple system for $v \equiv 9 \pmod{12}$ is proven. Note that, from Definition 12, each cyclic $BTS(v)$ is generated from triples having orbits of size v .Now, it can be said that the set of triples of R_0 in Table 2 is a starter of cyclic $BTS(9)$,Through the next theorem, the necessary conditions for the existence of a cyclic $BTS(12n+9)$ is provided.

Theorem 13. For any $n \geq 0$, there exists a cyclic $BTS(12n+9)$.

Proof. To prove this theorem, it suffices to construct a starter of cyclic $BTS(12n+9)$. Based on Theorem 7, it can be concluded that for all $v \equiv 9 \pmod{12}$, there exists a cyclic $\left(v, \frac{v-1}{2}, 2\right)$-$NRCS$. Suppose that $\mathcal{S} = \{C_{(6n+4)_1}, C_{(6n+4)_2}\}$ is a starter of cyclic $(12n+9, 6n+4, 2)$-$NRCS$. Then, based on Definition 12, $\mathcal{S}$ forms a near-two-factor with focus zero; this implies that the vertex set of $\mathcal{S}$ covers each nonzero element of $\mathbb{Z}_{12n+9}$ exactly once. However, since $\mathcal{S}$ contains two $(6n+4)$-cycles, it follows that the edge $\mathcal{S}$ set of contains $(12n+8)$ distinct edges such that each nonzero element of $\mathbb{Z}_{12n+9}$ is contained in exactly two edges.

According to Definition 12, $BTS(12n+9)$ is a $(12n+9)\times(12n+8)$ array of triples, $(12n+8)$ triples are needed in order to construct a starter of cyclic $BTS(12n+9)$.Let $\mathcal{T}$ be a set of triples that is obtained by appending the endpoints of each edge in $\mathcal{S}$ with the vertex zero. Then, $\mathcal{T}$ contains $(12n+8)$ triples among which the object zero is contained in each triple and every object other than zero is contained in exactly two triples. Now, $\mathcal{T}$ satisfies the conditions to be a starter of cyclic $BTS(12n+9)$. ϒ

The following subsection proposes a construction method for a cyclic Butterfly triple system.

Cycle Star Construction Method

Finding the starter set is the foundation in the construction process. Here, a construction method called cycle star construction method is developed for constructing a starter of the cyclic Butterfly triple system.

This method is divided into the following four steps:

Step 1. Construct the starter of cyclic $\left(v,\frac{v-1}{2},2\right)$-$NRCS$.

Step 2. Construct the starter of cyclic $\left(\frac{v-1}{2}\right)$ star decomposition of $2K_v$.

Step 3. Combine the similar vertices from Steps 1 and 2.

Step 4. Partition the graphs of Step 3 into triples.

Example 14 demonstrates the step-by-step construction of cyclic $BTS(9)$ by using the cycle star construction method.

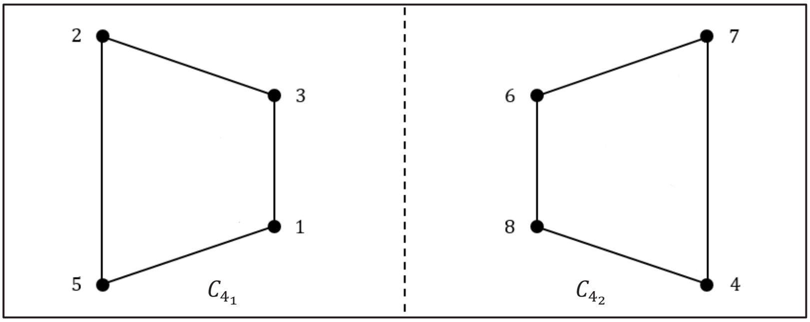
Example 14. The construction for a starter of cyclic $BTS(9)$.

Step 1. Construct the starter of cyclic $(9,4,2)$-$NRCS$.

Based on the construction in Equation (1) ,the set $\mathcal{S}=\{C_{4_1},C_{4_2}\}$ is a starter of the cyclic Butterfly 4-cycle system of $2K_9$, where $C_{4_1}=(2,5,1,3)$ and $C_{4_2}=(7,4,8,6)$ as can be seen in Figure 3.

Figure 3

The Construction of C_{4_1} and C_{4_2} in $2K_9$

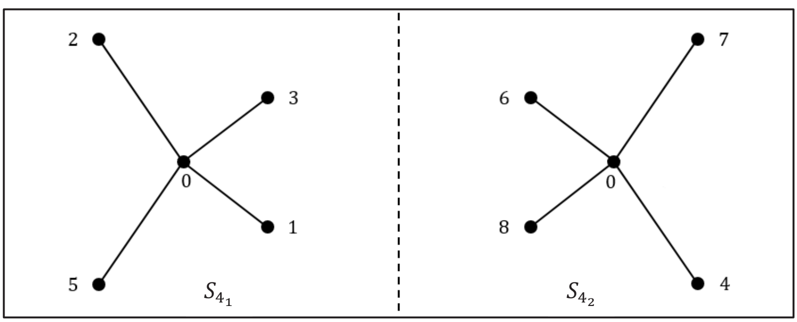


Step 2. Construct the starter of cyclic 4-star decomposition of $2K_9$.

From Equation (2), the set $\mathcal{F} = \{S_{4_1}, S_{4_2}\}$ is a starter for cyclic 4-star decomposition of $2K_9$, where $S_{4_1} = (0;\ 2, 5, 1, 3)$ and $S_{4_2} = (0;\ 7, 4, 8, 6)$, as shown in Figure 4.

Figure 4

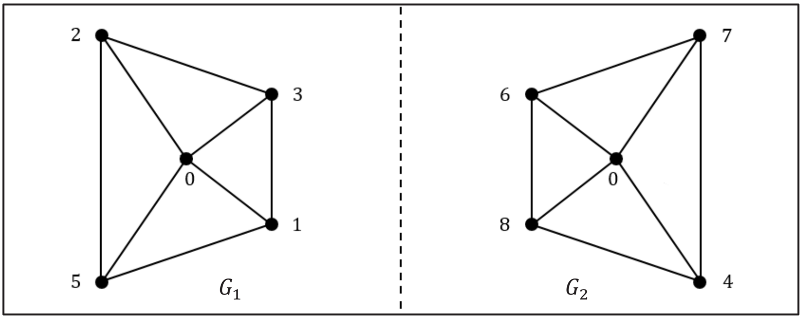
The Construction of S_{4_1} and S_{4_2} in $2K_9$



Step 3. Combine the similar vertices from Steps 1 and 2.
Let $\mathcal{G} = \{G_1, G_2\}$ be a set of graphs obtained by combining the similar vertices of the cycle C_{4_i} and the star S_{4_i} for $= 1, 2,$ as shown in Figure 5.

Figure 5

The Construction of G_1 and G_2 in $4K_9$

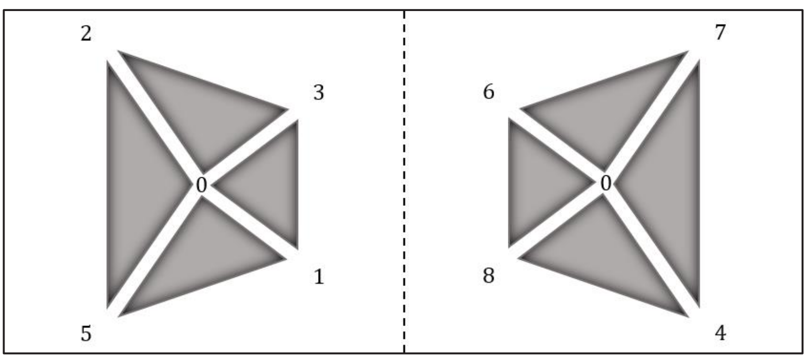


From Steps 1 and 2, it can be seen that the list of differences from $\mathcal{S} = \{C_{4_1}, C_{4_2}\}$ and the list of differences from $\mathcal{F} = \{S_{4_1}, S_{4_2}\}$ cover each nonzero element of $\mathbb{Z}_5$ exactly twice. Therefore, it can be said that the list of differences from $\mathcal{G}$ covers each nonzero element of $\mathbb{Z}_5$ exactly four times.

Step 4. Partition the graphs of Step 3 into triples.
The following figure shows how to partition the set of graphs $\mathcal{G} = \{G_1, G_2\}$ in Step 3 into triples.

Figure 6

Partition the Graphs G_1 and G_2 and into Triples



Let $\mathcal{T}$ be a set of triples obtained from the partitioning of the graphs G_1 and G_2. As shown in Figure 6, every edge in the set of cycles $\mathcal{S} = \{C_{4_1}, C_{4_2}\}$

is contained in exactly one triple of $\mathcal{T}$,while every edge in the set of stars $\mathcal{F} = \{S_{4_1}, S_{4_2}\}$is contained in exactly two triples of $\mathcal{T}$.Therefore, it can be concluded that the list of differences from $\mathcal{T}$ covers each nonzero element of $\mathbb{Z}_5$ exactly six times.

Now, all triples of cyclic $BTS(9)$ can be generated from the set of triples $\mathcal{T}$ by repeatedly adding1 modulo 15,as illustrated previously in Table 2.

Any collection $\mathcal{T}$ of triples of $\mathbb{Z}_v$ is called a λ-fold triple system if each pair of distinct elements of $\mathbb{Z}_v$is contained in exactly λ triples of $\mathcal{T}$.Then, since a Butterfly triple system of order v is an array of $v \times (v-1)$ triples of $\mathbb{Z}_v$ satisfying certain specified conditions, this naturally leads to a question on whether there is a relationship between the Butterfly triple system and λ-fold triple system. This question will be answered in the next results.

Theorem 15. There exists a cyclic 6 -fold Butterfly triple system for $v \equiv 9$ (mod 12)

Proof. Let $v = 12n + 9$where $n \geq 0$.This theorem is proven by using the cycle star construction method for constructing a starter of cyclic $BTS(12n+9)$ according to the following steps.

Step 1. Construct the starter of cyclic $(12n+9, 6n+4, 2)$-$NRCS$.
Suppose that $\mathcal{S} = \{C_{(6n+4)_1}, C_{(6n+4)_2}\}$ is a set of $(6n+4)$-cycles of $2K_{12n+9}$,which is defined in Equation (1).Then,$\mathcal{S}$is a starter of cyclic $(12n+9, 6n+4, 2)$-$NRCS$ such that $\mathcal{S}$ forms a near-two-factor of $2K_{12n+9}$ with focus zero and the list of differences from $\mathcal{S}$ covers $\mathbb{Z}_{6n+5} - \{0\}$ exactly twice.

Step 2. Construct the starter of cyclic $(6n+4)$-star decomposition of $2K_{12n+9}$.
Suppose that $\mathcal{F} = \{S_{(6n+4)_1}, S_{(6n+4)_2}\}$ is a set of $(6n+4)$ -stars of $2K_{12n+9}$,which is defined in Equation (2)Then, $\mathcal{F}$ is a starter of cyclic $(6n+4)$ -star decomposition of $2K_{12n+9}$ such that $V(S_{(6n+4)_1}) \cup V(S_{(6n+4)_2}) = \mathbb{Z}_{12n+9}$ and the list of differences from $\mathcal{F}$ covers $\mathbb{Z}_{6n+5} - \{0\}$ exactly twice.

Step 3. Combine the similar vertices from Steps 1 and 2.
Let $\mathcal{G} = \{G_1, G_2\}$ be a set of graphs obtained by combining the similar vertices of the cycle $C_{(6n+4)_i}$ and the star $S_{(6n+4)_i}$ for $= 1, 2$.

Step 4. Partition the graphs of Step 3 into triples.
Let $\mathcal{T}$ be a set of triples obtained from the partitioning of the graphs G_1 and G_2.Then, each triple $\mathcal{T}$ in is formed by joining an edge in $C_{(6n+4)_i}$ with two edges in $S_{(6n+4)_i}$ for $= 1, 2$.Therefore, the list of differences from $\mathcal{T}$ can be written as:

$$D(\mathcal{T}) = D(\mathcal{S}) \underline{\cup} D(\mathcal{F}) \underline{\cup} D(\mathcal{F}). \qquad (4)$$

However, from Steps 1 and 2, there are $D(\mathcal{S})$ and $D(\mathcal{F})$ that cover each nonzero element in $\mathbb{Z}_{6n+5}$ exactly twice. Therefore, from Equation 4, it follows that $D(\mathcal{T})$ covers each nonzero element in $\mathbb{Z}_{6n+5}$ exactly six times. Based on Lemma 9, $\mathcal{T}$ is a starter of the cyclic 6-fold triple system Υ

Algorithm for the Starter of Cyclic Butterfly Triple System

Based on the cycle star construction method, an algorithm is formulated for generating the starter of cyclic $BTS(12n + 9)$.

Reviewing the constructions of a starter of simple cyclic $(12n + 9, 6n + 4, 2)$-$NRCS$, as shown in Figures 1 and 2, it is noted that both constructions contain two components. Furthermore, in a similar representation, the algorithm for the starter of cyclic $BTS(12n + 9)$ is partitioned into two disjoint sets: the starter triples from the left wing and the starter triples from the right wing.

Case 1. n is odd.
Figure 7 illustrates the result of applying the cycle star construction method for constructing the starter of cyclic $BTS(12n + 9)$ when n is odd.

As shown in Figure 7, the generated triples from the left-wing partition can be expressed as a union of sets of the form:
$\mathcal{A}_1 = \bigcup_{i=1}^{7} A_{1,i}$,

where
$A_{1,1} = \{\{0, 4n + 5, 1\}, \{0, 10n + 8, 10n + 6\}, \{0, 4n + 3, 2\}\}$,
$A_{1,2} = \{\{0, 4i - 2, 12n - 4i + 9\} \mid \ 1 \le i \le 2n + 1\}$,

$A_{1,3} = \{\{0, 12n - 4i + 9, 4i + 2\} \mid 1 \le i \le 2n\}$,
$A_{1,4} = \left\{\{0, 4i - 3, 12n - 4i + 10\} \mid 1 \le i \le \frac{n+1}{2}\right\}$,
$A_{1,5} = \left\{\{0, 12n - 4i + 10, 4i + 1\} \mid 1 \le i \le \frac{n-1}{2}\right\}$,
$A_{1,6} = \left\{\{0, 10n - 4i + 10, 2n + 4i + 1\} \mid 1 \le i \le \frac{n+1}{2}\right\}$,
$A_{1,7} = \left\{\{0, 2n + 4i + 1, 10n - 4i + 6\} \mid 1 \le i \le \frac{n-1}{2}\right\}$.

Meanwhile, the generated triples from the right wing are expressed as a union of sets as follows:

$\mathcal{A}_2 = \bigcup_{i=1}^{7} A_{2,i}$,

where

$A_{2,1} = \{\{0, 8n + 4, 12n + 8\}, \{0, 2n + 1, 2n + 3\}, \{0, 8n + 6, 12n + 7\}\}$,
$A_{2,2} = \{\{0, 12n - 4i + 11, 4i\} \mid 1 \le i \le 2n + 1\}$,
$A_{2,3} = \{\{0, 4i, 12n - 4i + 7\} \mid 1 \le i \le 2n\}$,
$A_{2,4} = \left\{\{0, 12n - 4i + 12, 4i - 1\} \mid 1 \le i \le \frac{n+1}{2}\right\}$,
$A_{2,5} = \left\{\{0, 4i - 1, 12n - 4i + 8\} \mid 1 \le i \le \frac{n-1}{2}\right\}$,
$A_{2,6} = \left\{\{0, 2n + 4i - 1, 10n - 4i + 8\} \mid 1 \le i \le \frac{n+1}{2}\right\}$,
$A_{2,7} = \left\{\{0, 10n - 4i + 8, 2n + 4i + 3\} \mid 1 \le i \le \frac{n-1}{2}\right\}$.

Simply, it can be said that for all : $1 \le i \le 7$:

$$A_{2,i} = \left\{\{0, (12n + 9) - x, (12n + 9) - y\} \mid \{0, x, y\} \in A_{1,i}\right\}. \quad (5)$$

Consequently, the starter of cyclic Butterfly triple system of order $12n + 9$, when n is odd, is formed by $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$.

Case 2. n is even.
In the same manner as Case 1, the starter triples of cyclic $BTS(12n + 9)$ where n is even can be described as a union of two sets $\mathcal{A}_1 \cup \mathcal{A}_2$. Therefore, each set is represented as one side of the butterfly wings, in which the generated triples from the left wing are expressed as follows:

$\mathcal{A}_1 = \bigcup_{i=1}^{7} A_{1,i}$,

where

$$A_{1,1} = \{\{0, 4n+5, 1\}, \{0, 2n+1, 2n+3\}, \{0, 4n+3, 2\}\},$$
$$A_{1,2} = \{\{0, 4i-2, 12n-4i+9\} \mid 1 \le i \le 2n+1\},$$
$$A_{1,3} = \{\{0, 12n-4i+9, 4i+2\} \mid 1 \le i \le 2n\},$$
$$A_{1,4} = \left\{\{0, 4i-3, 12n-4i+10\} \mid 1 \le i \le \frac{n}{2}\right\},$$
$$A_{1,5} = \left\{\{0, 12n-4i+10, 4i+1\} \mid 1 \le i \le \frac{n}{2}\right\},$$
$$A_{1,6} = \left\{\{0, 2n+4i-1, 10n-4i+8\} \mid 1 \le i \le \frac{n}{2}\right\},$$
$$A_{1,7} = \left\{\{0, 10n-4i+8, 2n+4i+3\} \mid 1 \le i \le \frac{n}{2}\right\}.$$

Then, the generated triples from the right wing are expressed as follows:

$\mathcal{A}_2 = \bigcup_{i=1}^{7} A_{2,i}$,

where $A_{2,i}$ is constructed using Equation 5.

Example 16. The starter of cyclic $BTS(21)$ can be listed by choosing $n = 1$ in the algorithm for the starter of cyclic $BTS(12n+9)$ when n is odd. Now, the starter of cyclic $BTS(21)$ can be represented as a butterfly wing as follows:

The left wing is
where

$$A_{1,1} = \{\{0, 9, 1\}, \{0, 18, 16\}, \{0, 7, 2\}\},$$
$$A_{1,2} = \{\{0, 2, 17\}, \{0, 6, 13\}, \{0, 10, 9\}\},$$
$$A_{1,3} = \{\{0, 17, 6\}, \{0, 13, 10\}\},$$
$$A_{1,4} = \{\{0, 1, 18\}\},$$
$$A_{1,5} = \emptyset,$$
$$A_{1,6} = \{\{0, 16, 7\}\},$$
$$A_{1,7} = \emptyset.$$

The right wing is $\mathcal{A}_2 = \bigcup_{i=1}^{7} A_{2,i}$,
where
$$A_{2,1} = \{\{0, 12, 20\}, \{0, 3, 5\}, \{0, 14, 19\}\},$$

$A_{2,2} = \{\{0, 19, 4\}, \{0, 15, 8\}, \{0, 11, 12\}\}$,
$A_{2,3} = \{\{0, 4, 15\}, \{0, 8, 11\}\}$,
$A_{2,4} = \{\{0, 20, 3\}\}$,
$A_{2,5} = \emptyset$,
$A_{2,6} = \{\{0, 5, 14\}\}$,
$A_{2,7} = \emptyset$.

However, a cyclic $BTS(21)$ is constructed by repeatedly adding 1 21 modulo to a starter of cyclic . As noted above, the starter of cyclic $BTS(21)$consists of 20 distinct triples in which the vertex zero meets every other vertex exactly twice. Furthermore, for any triple $\{0, x, y\}$ contained in the first half of the wings, the triple$\{0, 21 - x, 21 - y\}$ is contained in the second half of the wings.

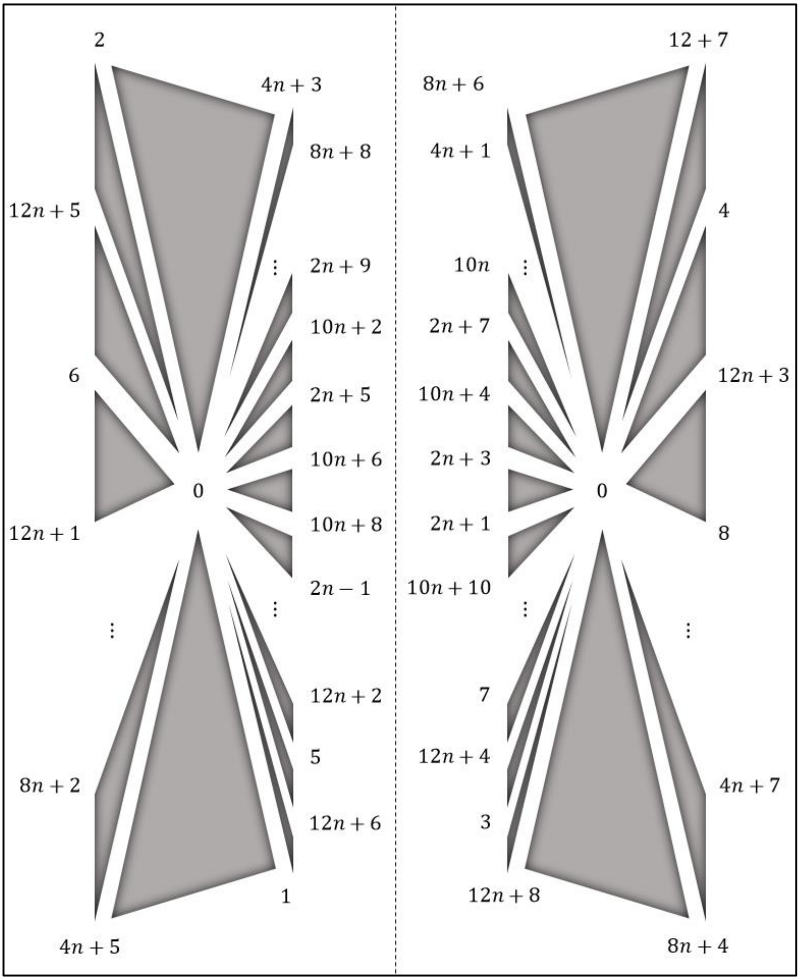
Simply said, the starter of the cyclic Butterfly triple system can be generated by the following algorithm 1.

Algorithm 1: Starter of Cyclic $BTS(12n + 9)$

1: Input: A number n
2: $\mathcal{S} \coloneqq$ a starter of cyclic $(12n + 9, 6n + 4, 2)$-$NRCS$
3: $c_{i,j} \coloneqq$ a vertex in $\mathcal{S}$ for $1 \le i \le 2,\ 1 \le j \le 6n + 4$
4: $A_1 \coloneqq \left\{\{0, c_{1,j}, c_{1,j+1}\} \middle|\ 1 \le j \le 6n + 4,\ c_{1,6n+5} = c_{1,1}\right\}$
5: $A_2 \coloneqq \left\{\{0, c_{2,j}, c_{2,j+1}\} \middle|\ 1 \le j \le 6n + 4,\ c_{2,6n+5} = c_{2,1}\right\}$
6: Output: $A_1 \cup A_2$

Figure 7

Starter of Cyclic $BTS(12n+9)$ *when is Odd*



CONCLUSION

This study proposed a new triple system, called a Butterfly triple system, to solve the problem of decomposing triples of $\mathbb{Z}_v$ into cyclic triple systems for $v \equiv 9 \pmod{12}$. Then, the construction of cyclic near-resolvable $\left(\frac{v-1}{2}\right)$ cycle system $2K_v$ of and cyclic $\left(\frac{v-1}{2}\right)$-star decomposition of $2K_v$ were employed to construct the starter of the cyclic Butterfly triple system, which was later called the cycle star construction method. This study can be extended to contribute toward solving the problem of decomposing all triples of $\mathbb{Z}_v$ into cyclic triple systems for all odd v.

ACKNOWLEDGMENT

This research was supported by the Ministry of Education (MOE) through the Fundamental Research Scheme (FRGS/1/2020/STG06/UUM/01/1). The authors would also like to thank anonymous referees for their useful suggestions.

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