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Pricing variance swaps under stochastic volatility and stochastic interest rate

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ABSTRACT

In this paper, we investigate the effects of imposing stochastic interest rate driven by the Cox–Ingersoll–Ross process along with the Heston stochastic volatility model for pricing variance swaps with discrete sampling times. A dimension reduction mechanism based on the framework of Little and Pant (2001) is applied which later reduces to solving two three-dimensional partial differential equations. A semi-closed form solution to the fair delivery price of a variance swap is obtained via the derivation of characteristic functions. Practical implementation of this hybrid model is demonstrated through numerical simulations.

1. Introduction

Volatility is a measure of the price fluctuation of a financial instrument over time. However, volatility/variance has become a class of trading assets in its own right in the past twenty years. In late 1990s, Wall Street firms started trading variance swaps, forward contracts written on the realized variance. Basically, variance swaps are categorized under volatility derivatives which are financial derivatives and their values depend on the future levels of volatility. According to Demeterfi et al. [8], volatility derivatives are traded for decision-making between long or short positions, trading spreads between realized and implied volatility, and hedging against volatility risks. The utmost advantage of volatility derivatives is their capability in providing direct exposure towards the assets volatility without being burdened with the hassles of continuous delta-hedging. The measures of volatility involved can be categorized into three main areas which are historical volatility, implied volatility and model-based volatility. Historical volatility is mainly related to previous standard deviation of financial returns involving a specified time period. An example of a volatility derivative written on this historical volatility measure is the futures on realized variance. The implied-volatility ascertain the volatility by matching volatilities from the market and some specific pricing model. The VIX estimate this type of volatility measure of the S&P 500 index. Finally, the model-based volatility are defined in the class of stochastic volatility models such as [17,27] and others.

Researchers working in the field concerning volatility derivatives have been focusing on developing suitable methods for estimating values of variance swaps. In addition, incorporation of stochastic volatility into the models of pricing and

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hedging variance swaps have also been a trend in the recent literature. For example, based on the Heston stochastic volatility model, Grunbichler and Longstaff [14] developed a pricing model for options on variance. One important finding was the contrast characteristics between volatility derivatives and usual equity options on traded assets. In 1998, Carr and Madan [5] combined static replication using options with dynamic trading in the futures to price and hedge certain volatility contracts without specifying the volatility process. The principle assumptions were continuous trading and continuous semi-martingale price processes for the future prices. Selection of a payoff function which diminished the path dependence property ensured that the investor's joint perception regarding volatility and price was also taken into consideration. Further, Demeterfi et al. [8] proved that a variance swap could be reproduced via a portfolio of standard options. The requirements specified were continuity of exercise prices for the options and continuous sampling times for the variance swaps. However, it was later noted by Heston and Nandi [18] that specifying the mean reverting square root process has the disadvantage of unobservable underlyings. Thus, the latter proposed a user friendly model by working on the discrete-time GARCH volatility process with parametric specifications. This model had the advantage of real market practicability, as well as the capability to hedge various volatility derivatives using only a single asset. In 2007, Elliott et al. [12] constructed a continuous-time Markovian-regulated version of the Heston stochastic volatility model to distinguish the states of a business cycle. Analytical formulas were obtained using the regime-switching Esscher transform and price comparisons were made between models with and without switching regimes. Their results showed that the prices of variance swaps implied by the regime-switching Heston stochastic volatility model were significantly higher than those without switching regimes. One important characteristic shared among these researchers was the assumption of continuous sampling time which is actually an inclination with the discrete sampling reality in financial markets. In fact, options of discretely variance swaps were mis-valued when the continuous sampling were used as approximations, because these continuous approximations produce non-negligible inaccuracies in certain sampling periods as discussed by Bernard and Cui [1], Elliott and Lian [10], Little and Pant [22], and Zhu and Lian [30]. It is worth mentioning that stochastic models have not only been used in quantitative finance in terms of pricing financial derivatives, but also gained popularity in other field, see, for example, [23,24,28].

In addition to previously mentioned analytic approaches, Little and Pant [22] explored the finite difference method in the numerical approaches via dimension-reduction approach and contributed to high efficiency and accuracy for discretely sampled variance swaps. In addition, Windcliff et al. [29] investigated the effects of employing the partial integro differential equation on constant volatility, local volatility and jump diffusion-based volatility products. Their delta-gamma hedging models accomplishes less liability and is effective for discontinuous market traits and ordinary counter-party writing instruments. The work of Little and Pant was extended by Zhu and Lian [30] through incorporating Heston two-factor stochastic volatility for pricing discretely sampled variance swaps. Levels of validity for short periods when using the continuous-time sampling were provided through significant errors, along with analytical hedging derivations and numerical simulations. A recent study was conducted by Bernard and Cui [1] on analytical and asymptotic results for discrete sampling variance swaps with three different stochastic volatility models.

Despite the popularity of pricing discrete variance swaps in the literature reviewed above, an approach for determining the price of discrete variance swaps based on stochastic volatility and stochastic interest rate has not yet seen. The assumption of constant interest rates in the existing literatures in pricing variance swaps were unrealistic in modeling the real market phenomena. Thus, the novelty of our work in the current paper is to price discrete variance swaps by incorporating stochastic interest rate along with stochastic volatility. In this way, we can not only reduce the inaccuracies resulting from continuous sampling time for pricing variance swaps, but also provide a better market characterization with stochastic interest rate.

In the past three decades, many authors have considered modeling stochastic interest rate and its applications in pricing financial derivatives using stochastic approaches. Generally, the modeling trend can be seen as developing from unobservable rates such as spot rates to market rates regularly practiced by financial institutions. Elliott and Siu [11] pointed out that the stochastic interest rate models should be capable of providing a practical realization of the fluctuation property, as well as adequate tractability. They derived exponential affine form of bond prices with elements of continuous-time Markov chain using enlarged filtration and semi-martingale decompositions. In addition, Grzelak and Oosterlee [15] examined correlation issues of European products pricing with the Heston–Hull–White and Heston–CIR hybrid models. Recently, Kim et al. [20] showed that incorporation of stochastic interest rates into a stochastic volatility model gave better results compared with the constant interest rate case in any maturity. They proposed a model which was a combination of the multi-scale stochastic volatility model from [13] and the Hull–White interest rate model. The call option price approximation for this mixed model was obtained via derivation of the leading order and the first order correction prices using Fouque's multiscale expansion method and operator specifications for the correction terms. In a quite recent paper [26], Shen and Siu investigated the effects of stochastic interest rates and stochastic regime-switching volatility for the pricing of variance swaps. However, only continuous sampling approximation in an integral form were formulated for variance swap rates in these research.

In this paper, a hybridization of the Heston stochastic volatility model and the CIR stochastic interest rate model is employed to investigate its effects on the pricing rates of variance swap with discrete sampling. This hybrid model extends the work of Zhu and Lian [30] where stochastic interest rates were ignored. A semi-closed form solution to the fair delivery price of a discretely sampled variance swap is obtained via the dimension reduction technique and derivation of characteristic functions. Comparison between the results obtained from our numerical analysis with those results from other existing models using constant interest rates indicates that our model fills the gap left in the literature on pricing variance swaps. As for market practitioners, the benefits will be achieved in terms of computational efficiency and developing new pragmatic mechanisms for their trading activities.

2. Variance swaps and our model

In this section, we introduce the Heston–CIR hybrid model, variance swaps and relevant concepts. We also outline our solution approach to the valuation of variance swaps.

2.1. The realized variance and variance swaps

Roughly speaking, the *realized variance* (RV) is the sum of squared returns. It provides a relatively accurate measure of volatility which is useful for many purposes, including volatility forecasting and forecast evaluation. Let $S(t)$ be the price of an asset over a finite time horizon $[0, T]$. Suppose that $S(t)$ is observed at N times during the period $[0, T]$: $0 \le t_1 < t_2 < \cdots < t_N \le T$. A typical formula for the measure of RV is given by

$$RV = \frac{AF}{N}\sum_{j=1}^{N}\left(\frac{S(t_j)-S(t_{j-1})}{S(t_{j-1})}\right)^2 \times 100^2, \tag{2.1}$$

where AF is the annualized factor converting this expression to an annualized variance, depending on the sampling frequency. If the sampling frequency is every trading day, then $AF = 252$, assuming that there are 252 trading days in one year; if every week then $AF = 52$; if every month then $AF = 12$. In this paper, we assume equally spaced discrete observations and take $AF = \frac{1}{\Delta t} = \frac{N}{T}$ to express RV in terms of basis point per unit time.

A *variance swap* is a forward contract on the future realized variance of the returns of the specified asset. The long position of a variance swap pays a fixed delivery price at the expiration and receives the floating amounts of annualized realized variance, whereas the short position is the opposite. Suppose that the mature time of a variance swap on an asset with price $S(t)$ is T. Then, the payoff of the long position of a variance swap at the maturity T is

$$(RV - K) \times L, \tag{2.2}$$

where K is the annualized delivery price of the variance swap and L is the notional amount of the swap in dollars per annualized volatility point squared (also known as the face amount).

2.2. The Heston-–CIR hybrid model

As revealed by many empirical studies [16,19], the classical Black–Scholes model in [2] may fail to reflect certain features of financial markets due to some unrealistic assumptions including the constant volatility and constant interest rate assumptions. To remedy these drawbacks of the Back–Scholes model, many models have been proposed by academic researchers and practitioners to incorporate stochastic volatility, jump diffusion and stochastic interest rate [7,9,27]. Among stochastic volatility models, the one proposed by Heston [17] has received a lot of attentions, since it gives satisfactory description of the underlying asset dynamics [10,12]. Recently, Zhu and Lian [30,31] used Heston's model to derive a closed form exact solution to the price of variance swaps. In this paper, we will move a step further and apply certain hybridization of the Heston stochastic volatility model in [17] and the Cox–Ingersoll–Ross (CIR) interest rate model in [7] to the valuation of variance swaps. Note that Heston–CIR hybrid models have been discussed and applied to study the affine approximation pricing techniques with correlations, the convergence of approximated prices using discretization methods and the pricing of American options, refer to [6,15,21]. The Heston–CIR hybrid model that we shall use in the sequel can be described as follows

$$\begin{cases} dS(t) = \mu S(t)dt + \sqrt{\nu(t)}S(t)dW_1(t), \\ d\nu(t) = \kappa(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)}dW_2(t), \\ dr(t) = \alpha(\beta - r(t))dt + \eta\sqrt{r(t)}dW_3(t), \end{cases} \tag{2.3}$$

where $r(t)$ is the stochastic instantaneous interest rate in which α determines the speed of mean reversion for the interest rate process, β is the interest rate term structure and η controls the volatility of the interest rate. In the stochastic instantaneous variance process $\nu(t)$, κ is its mean-reverting speed parameter, θ is its long-term mean and σ is its volatility. In order to ensure that the square root processes are always positive, it is required that $2\kappa\theta \ge \sigma^2$ and $2\alpha\beta \ge \eta^2$ respectively, [7,17]. Throughout this paper, we assume that correlations involved in the above model are given by $(dW_1(t), dW_2(t)) = \rho dt$, $(dW_1(t), dW_3(t)) = 0$ and $(dW_2(t), dW_3(t)) = 0$, where ρ is a constant with $-1 \le \rho \le 1$.

According to Girsanov's theorem, there exists a risk-neutral probability measure $\mathbb{Q}$ equivalent to the real world probability measure $\mathbb{P}$ such that under $\mathbb{Q}$ system (2.3) is transformed into the form of

$$\begin{cases} dS(t) = r(t)S(t)dt + \sqrt{\nu(t)}S(t)d\widetilde{W}_1(t), \\ d\nu(t) = \kappa^*(\theta^* - \nu(t))dt + \sigma\sqrt{\nu(t)}d\widetilde{W}_2(t), \\ dr(t) = \alpha^*(\beta^* - r(t))dt + \eta\sqrt{r(t)}d\widetilde{W}_3(t), \end{cases} \tag{2.4}$$

where $\kappa^* = \kappa + \lambda_1$, $\theta^* = \frac{\kappa\theta}{\kappa+\lambda_1}$, $\alpha^* = \alpha + \lambda_2$ and $\beta^* = \frac{\alpha\beta}{\alpha+\lambda_2}$ are the risk-neutral parameters, $\widetilde{W}_i(t)$ $(1 \le i \le 3)$ is a Brownian motion under $\mathbb{Q}$. Here, λ_j $(j = 1, 2)$ is the premium of volatility or interest rate risk.

Applying the Cholesky decomposition of a correlation matrix, (2.4) can be re-written as

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \kappa^*(\theta^* - \nu(t)) \\ \alpha^*(\beta^* - r(t)) \end{bmatrix} dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \tag{2.5}$$

with

$$\Sigma = \begin{bmatrix} \sqrt{\nu(t)} & 0 & 0 \\ 0 & \sigma\sqrt{\nu(t)} & 0 \\ 0 & 0 & \eta\sqrt{r(t)} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

such that

$$CC^\top = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $dW_1^*(t)$, $dW_2^*(t)$ and $dW_3^*(t)$ are mutually independent under $\mathbb{Q}$ satisfying

$$\begin{bmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \\ d\widetilde{W}_3(t) \end{bmatrix} = C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}.$$

3. Pricing variance swaps under the Heston–CIR hybrid model

In this section, we will derive a semi-closed form solution for the delivery price of a variance swap in a Heston–CIR hybrid model.

3.1. Outline of the solution approach

In the risk-neutral world, the value of a variance swap at time $0 \le t \le T$ is the expected present value of the future payoff which is given in (2.2). Mathematically, it can be presented as,

$$\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r(s)ds}(RV - K) \times L]. \tag{3.1}$$

This value should be zero at $t = 0$ to ensure that both parties of the contract are fairly treated. To determine the value of K, we need to change $\mathbb{Q}$ to the forward measure $\mathbb{Q}^T$ in the above expression. Since the price of a T-maturity zero-coupon bond at $t = 0$ is given by $P(0,T) = \mathbb{E}_0^{\mathbb{Q}}[e^{-\int_0^T r(s)ds}]$, then

$$\mathbb{E}_0^{\mathbb{Q}}[e^{-\int_0^T r(s)ds}(RV - K) \times L] = P(0,T)\mathbb{E}_0^T[(RV - K) \times L], \tag{3.2}$$

where $\mathbb{E}_0^T[\cdot]$ denotes the expectation with respect to $\mathbb{Q}^T$ at $t = 0$. Thus, the fair delivery price of the variance swap is defined as $K = \mathbb{E}_0^T[RV]$, and the problem of obtaining the fair delivery price of a variance swap is reduced to calculating N many expectations expressed in the form of

$$\mathbb{E}_0^T\left[\left(\frac{S(t_j) - S(t_{j-1})}{S(t_{j-1})}\right)^2\right] \tag{3.3}$$

for some fixed equal time period Δt, $t_0 = 0$ and $t_j = j\Delta t$ $(j = 1, \ldots, N)$. To this end, we need to consider two cases: $j = 1$ and $j > 1$, due to the difference in the calculation procedures. In the process of calculating this expectation, j, unless otherwise stated, is regarded as a constant. Hence, both t_j and t_{j-1} are regarded as known constants.

When $j > 1$, both $S(t_{j-1})$ and $S(t_j)$ are unknown at $t = 0$. Since

$$\mathbb{E}_0^T\left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1\right)^2\right] = \mathbb{E}_0^T\left[\mathbb{E}_{t_{j-1}}^T\left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1\right)^2\right]\right], \tag{3.4}$$

the calculation of expectation (3.3) is split into the calculation of E_{j-1}, which is defined by

$$E_{j-1} = \mathbb{E}_{t_{j-1}}^T\left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1\right)^2\right], \tag{3.5}$$

and $\mathbb{E}_0^T[E_{j-1}]$. Thus, our solution approach of finding the delivery price K of a variance swap can be split in two steps. In the first step, we shall calculate E_{j-1}, and in the second step, we need to calculate $\mathbb{E}_0^T[E_{j-1}]$ and K.

Note that the numeraire under $\mathbb{Q}$ is $N_{1,t} = e^{\int_0^t r(s)ds}$ and the numeraire under $\mathbb{Q}^T$ is $N_{2,t} = P(t,T)$ (refer [3] for more details). Applying the Radon–Nikodym derivative to these two numeraires, we can transform (2.5) under $\mathbb{Q}$ to the following system under $\mathbb{Q}^T$:

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \kappa^*(\theta^* - \nu(t)) \\ \alpha^*\beta^* - [\alpha^* + B(t,T)\eta^2]r(t) \end{bmatrix} dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \tag{3.6}$$

where

$$B(t,T) = \frac{2(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}} - 1)}{2\sqrt{(\alpha^*)^2 + 2\eta^2} + (\alpha^* + \sqrt{(\alpha^*)^2 + 2\eta^2})(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}} - 1)}.$$

3.2. The first step of computation

To calculate E_{j-1}, we consider a contingent claim $U_j(S(t), \nu(t), r(t), t)$ over $[t_{j-1}, t_j]$, whose payoff at expiry t_j is $(\frac{S(t_j)}{S(t_{j-1})} - 1)^2$. Applying standard techniques in the general asset valuation theory, we can obtain a PDE for U_j over $[t_{j-1}, t_j]$ as

$$\frac{\partial U_j}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 U_j}{\partial S^2} + \frac{1}{2}\sigma^2 \nu \frac{\partial^2 U_j}{\partial \nu^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 U_j}{\partial r^2} + \rho\sigma\nu S \frac{\partial^2 U_j}{\partial S \partial \nu} + rS\frac{\partial U_j}{\partial S} + \kappa^*(\theta^* - \nu)\frac{\partial U_j}{\partial \nu} + [\alpha^*\beta^* - (\alpha^* + B(t,T)\eta^2)r]\frac{\partial U_j}{\partial r} = 0 \tag{3.7}$$

with the terminal condition

$$U_j(S, \nu, r, t_j) = \left(\frac{S}{S(t_{j-1})} - 1\right)^2. \tag{3.8}$$

If the underlying asset follows the dynamic process (3.6), we can solve analytically PDE (3.7) with condition (3.8) by the generalized Fourier transform method. Its solution can be derived by the following general proposition.

Proposition 3.1. *If the underlying asset follows the dynamic process (3.6) and a European-style derivative written on this underlying asset has a payoff function $U(S, \nu, r, T) = H(S)$ at expiry T, then the solution to the associated PDE system of the derivative value*

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 \nu \frac{\partial^2 U}{\partial \nu^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 U}{\partial r^2} + \rho\sigma\nu S \frac{\partial^2 U}{\partial S \partial \nu} + rS\frac{\partial U}{\partial S} \\ \quad + \kappa^*(\theta^* - \nu)\frac{\partial U}{\partial \nu} + \left[\alpha^*\beta^* - (\alpha^* + B(t,T)\eta^2)r\right]\frac{\partial U}{\partial r} = 0, \\ U(S, \nu, r, T) = H(S) \end{cases} \tag{3.9}$$

can be expressed in semi-closed form as:

$$U(x, \nu, r, \tau) = \mathcal{F}^{-1}[e^{C(\omega,\tau) + D(\omega,\tau)\nu + E(\omega,\tau)r}\mathcal{F}[H(e^x)]], \tag{3.10}$$

in terms of the generalized Fourier transform (see [25]), where $x = \ln S$, $\tau = T - t$, $i = \sqrt{-1}$, ω is the Fourier transform variable,

$$\begin{cases} D(\omega, \tau) = \frac{a+b}{\sigma^2}\frac{1 - e^{b\tau}}{1 - ge^{b\tau}}, \\ a = \kappa^* - \rho\sigma\omega i, \quad b = \sqrt{a^2 + \sigma^2(\omega^2 + \omega i)}, \quad g = \frac{a+b}{a-b}, \end{cases} \tag{3.11}$$

$E(\omega, \tau)$ and $C(\omega, \tau)$ satisfy the following ODE system

$$\begin{cases} \frac{dE}{d\tau} = \frac{1}{2}\eta^2 E^2 - (\alpha^* + B(T-\tau, T)\eta^2)E + \omega i, \\ \frac{dC}{d\tau} = \kappa^*\theta^* D + \alpha^*\beta^* E, \end{cases} \tag{3.12}$$

with the initial conditions

$$C(\omega, 0) = 0, \quad E(\omega, 0) = 0. \tag{3.13}$$

The proof of this proposition is left in Appendix A.

Note that Proposition 3.1 is applicable to most derivatives whose payoffs depend on spot price S of underlying asset in the framework of the Heston–CIR hybrid model under our assumptions. However, in some cases, it is hard to handle the general Fourier transform. Next, we apply $H(S) = (\frac{S}{S(t_{j-1})} - 1)^2$ to Proposition 3.1, the Fourier inverse transform could be explicitly worked out and hence the solution to (3.7) can be written in an elegant form. The generalized Fourier transform can be described as

$$\mathcal{F}[e^{i\xi x}] = 2\pi\delta_\xi(\omega), \tag{3.14}$$

where ξ is any complex number and $\delta_\xi(\omega)$ is the generalized delta function satisfying

$$\int_{-\infty}^{\infty} \delta_\xi(x)\Phi(x)dx = \Phi(\xi). \tag{3.15}$$

For convenience, let $I = S(t_{j-1})$ and $x = \ln S$. We perform the generalized Fourier transform to the payoff function $H(e^x)$ with respect to x and derive

$$\mathcal{F}\left[\left(\frac{e^x}{I} - 1\right)^2\right] = 2\pi\left[\frac{\delta_{-2i}(\omega)}{I^2} - 2\frac{\delta_{-i}(\omega)}{I} + \delta_0(\omega)\right]. \tag{3.16}$$

Thus, the solution to (3.7) is given by

$$\begin{aligned} U_j(S, \nu, r, \tau) &= \mathcal{F}^{-1}\left[e^{C(\omega,\tau)+D(\omega,\tau)\nu+E(\omega,\tau)r} 2\pi\left[\frac{\delta_{-2i}(\omega)}{I^2} - 2\frac{\delta_{-i}(\omega)}{I} + \delta_0(\omega)\right]\right] \\ &= \int_{-\infty}^{\infty} e^{C(\omega,\tau)+D(\omega,\tau)\nu+E(\omega,\tau)r}\left[\frac{\delta_{-2i}(\omega)}{I^2} - 2\frac{\delta_{-i}(\omega)}{I} + \delta_0(\omega)\right] e^{x\omega i} d\omega \\ &= \frac{e^{2x}}{I^2} e^{\widetilde{C}(\tau)+\widetilde{D}(\tau)\nu+\widetilde{E}(\tau)r} - \frac{2e^x}{I} e^{\widehat{C}(\tau)+\widehat{E}(\tau)r} + 1, \end{aligned} \tag{3.17}$$

where $t_{j-1} \le t \le t_j$ and $\tau = t_j - t$. $\widetilde{C}(\tau)$, $\widetilde{D}(\tau)$ and $\widetilde{E}(\tau)$ are the notations for $C(-2i, \tau)$, $D(-2i, \tau)$ and $E(-2i, \tau)$ respectively, whereas $\widehat{C}(\tau)$ and $\widehat{E}(\tau)$ are equal to $C(-i, \tau)$ and $E(-i, \tau)$ respectively.

Finally, let $\tau = \Delta t$ in $U_j(S, \nu, r, \tau)$, and by noting that $\ln S(t_{j-1}) = \ln I(t)$ in (3.17), we obtain

$$E_{j-1} = e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)\nu(t_{j-1})+\widetilde{E}(\Delta t)r(t_{j-1})} - 2e^{\widehat{C}(\Delta t)+\widehat{E}(\Delta t)r(t_{j-1})} + 1. \tag{3.18}$$

3.3. The second step of computation

In this subsection, we calculate $\mathbb{E}_0^T[E_{j-1}]$ and K. Since E_{j-1} is an exponential function of the stochastic variables $\nu(t_{j-1})$ and $r(t_{j-1})$ in affine form, it is possible for us to carry out the expectation with a semi-closed form solution, by using the characteristic functions of $\nu(t_{j-1})$ and $r(t_{j-1})$. We assume that $\nu(t_{j-1})$ and $r(t_{j-1})$ are independent. Thus,

$$\mathbb{E}_0^T[E_{j-1}] = e^{\widetilde{C}(\Delta t)} \cdot \mathbb{E}_0^T[e^{\widetilde{D}(\Delta t)\nu(t_{j-1})}] \cdot \mathbb{E}_0^T[e^{\widetilde{E}(\Delta t)r(t_{j-1})}] - 2e^{\widehat{C}(\Delta t)} \cdot \mathbb{E}_0^T[e^{\widehat{E}(\Delta t)r(t_{j-1})}] + 1. \tag{3.19}$$

If we put

$$h(\phi, \nu, \tau) = \mathbb{E}_t^T[e^{\phi\nu(t+\tau)}] \tag{3.20}$$

and

$$f(\phi, r, \tau) = \mathbb{E}_t^T[e^{\phi r(t+\tau)}], \tag{3.21}$$

then we can express $\mathbb{E}_0^T[E_{j-1}]$ as follows:

$$\mathbb{E}_0^T[E_{j-1}] = e^{\widetilde{C}(\Delta t)} \cdot h(\widetilde{D}(\Delta t), \nu(0), t_{j-1}) \cdot f(\widetilde{E}(\Delta t), r(0), t_{j-1}) - 2e^{\widehat{C}(\Delta t)} \cdot f(\widehat{E}(\Delta t), r(0), t_{j-1}) + 1. \tag{3.22}$$

In Appendix B, we show how to derive expressions for f and h by solving the corresponding PDEs.

Now, by (2.1), we have

$$K = \mathbb{E}_0^T[RV] = \frac{100^2}{T}\sum_{j=1}^{N} \mathbb{E}_0^T[E_{j-1}]. \tag{3.23}$$

Using (3.22), the summation in (3.23) for the whole period of $j = 1$ to $j = N$ can now be carried out all the way except for the very first period with $j = 1$. We need to treat the case $j = 1$, separately, because in this case we have $t_{j-1} = 0$ and $S(t_{j-1}) = S(0)$ is a known value, instead of an unknown value of $S(t_{j-1})$ for any other cases with $j > 1$. Put

$$G(\nu(0), r(0)) = \mathbb{E}_0^T\left[\left(\frac{S(t_1)}{S(0)} - 1\right)^2\right], \quad G_j(\nu(0), r(0)) = \mathbb{E}_0^T[E_{j-1}]. \tag{3.24}$$

Table 1
Model parameters of the Heston–CIR hybrid model.

Parameters	S_0	ρ	v_0	θ^*	κ^*	σ	r_0	α^*	β^*	η	T
Values	1	−0.40	$(22.36\%)^2$	$(22.36\%)^2$	2	0.1	5%	1.2	5%	0.01	1

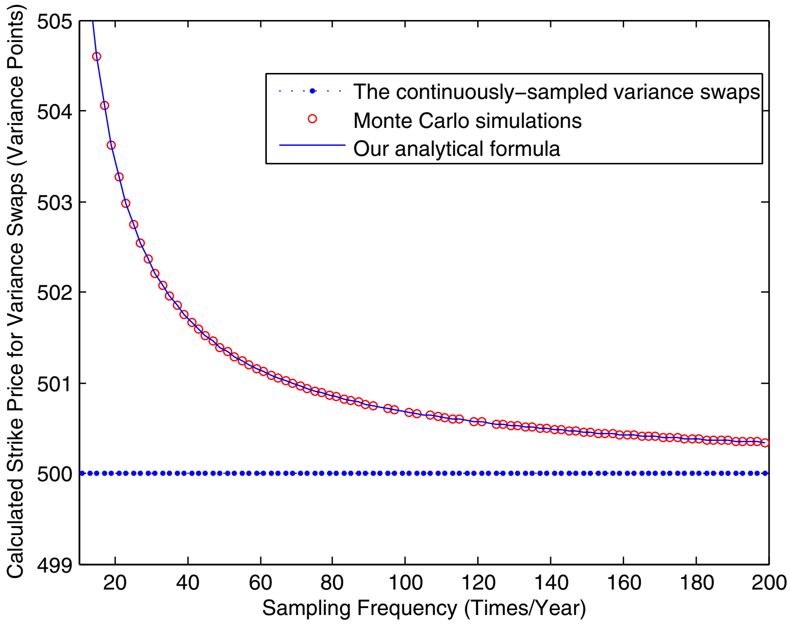

Fig. 1. The comparison of our formulae for variance swaps with MC simulations, with different sampling frequency. Model parameters are presented in Table 1. Time to maturity is 1 year. It shows that the results from our formulae (3.25), match up very well with these obtained from MC simulations which serve as benchmark values.

Then, $G(\nu(0), r(0))$ can be derived from Proposition 3.1 directly. Finally, we obtain the fair delivery price of a variance swap as:

$$K = \mathbb{E}_0^T[RV] = \frac{100^2}{T}\left[G(\nu(0), r(0)) + \sum_{j=2}^{N} G_j(\nu(0), r(0))\right]. \tag{3.25}$$

This formula is obtained by solving the associated PDEs in two steps. Since we have managed to express the solution of the associated PDEs, in both steps, in terms of simple and elementary functions, we are able to write the fair delivery price of a variance swap with discretely-sampled realized variance defined in its payoff in a simple and semi-closed form.

4. Numerical examples and simulation

In this section, we perform a numerical analysis for pricing variance swaps under our model and utilize our analytical pricing formula. We use the parameters in Table 1, unless otherwise stated, in all our numerical examples. This set of parameters for the hybrid Heston–CIR model was also adopted by Grzelak and Oosterlee [15].

4.1. Monte Carlo simulation

We firstly have implemented Monte Carlo (MC) simulations to obtain numerical results as references for comparisons. The stochastic processes of the model are discretized by using the simple Euler–Milstein scheme.

Fig. 1 shows the comparison between the numerical results obtained from our analytical formula (3.25), those from Monte Carlo simulations, the numerical calculation of the continuously-sampled realized variance. It is clearly seen that the results from our semi-closed form solution perfectly match the results from the MC simulations. For example, for the weekly-sampled variance swaps (the sampling frequency is 52 in the figure), the relative difference between numerical results obtained from formula (3.25) and the MC simulations is less than 0.05% already, when the number of paths reaches 200,000 in MC simulations. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the MC simulations towards our semi-closed form solution and hence to a certain extent providing a verification of our semi-closed form solutions.

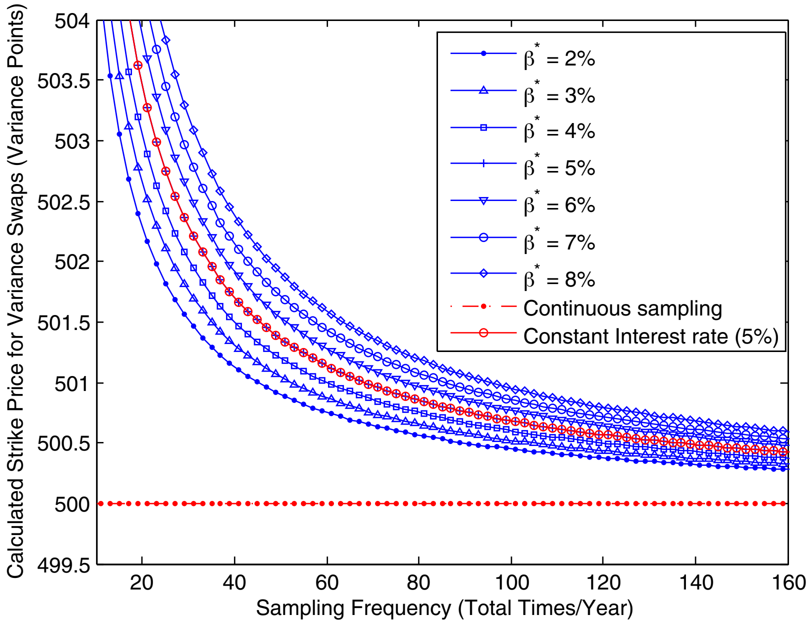

Fig. 2. The values of variance swaps with different β^* in the Heston–CIR hybrid model. Model parameters are presented in Table 1, except for β^* that can take different values as indicated in the figure. Time to maturity is 1 year.

4.2. Effect of stochastic interest rate

To test the effects of the stochastic interest rate, we now calculate the fair strike values of variance swaps with stochastic interest rate and deterministic interest rate. So we implement the analytical pricing formula (3.25) with the parameters tabulated in Table 1 (unless otherwise stated) to obtain numerical values of variance swaps with stochastic interest rate. For the variance swaps with constant deterministic interest rate, we implement the formula by Zhu and Lian [30]. Of course, since the analytical pricing formula (3.25) is derived based on a more general Heston–CIR hybrid model, we can obtain values of variance swaps with constant deterministic interest rate by setting $\alpha^* = 0,\ \beta^* = 0$ and $\eta = 0$. The time to maturity in all the numerical examples below is $T = 1$.

Fig. 2 depicts the fair strike values of variance swaps with different sampling frequencies, ranging from sampling 15 times in total per year to 160 times per year. We notice that with the increasing of sampling frequency, the values of discrete variance swaps are decreasing, converging to the continuous sampling counterpart. This is consistent with the convergence pattern of constant interest rate as shown in other research papers (see, for example [4,30]). We can also observe that, when the spot interest rate ($r_0 = 5\%$) is equal to the long-term interest rate (β^* in our notation), the values of variance swaps with stochastic interest rate coincide with the case of constant interest rate which remains unchanging as 5%. This implies that the parameters α^* and η have little effect on the values of variance swaps.

Finally, we can see that, when β^* is increasing, the values of variance swaps are increasing correspondingly. The implication is that the interest rate can impact and change the value of a variance swap, ignoring the effect of interest rate will result in mispricing. Because interest rate is changing and modeled by stochastic processes (such as a CIR process), working out the analytical pricing formula for discretely-sampled variance swaps in the Heston–CIR hybrid model can help pricing variance swaps more accurately.

5. Conclusion

This paper investigates the pricing of discrete variance swaps in the framework of stochastic interest rate and stochastic volatility. We solved the governing PDEs and derived an analytical pricing formula based on the Heston–CIR hybrid model. We compared the numerical results obtained from our pricing formula with those from Monte-Carlo simulations and showed they match up very well, providing a verification of the correctness of our pricing formula. With the availability of the analytical pricing formula, we also discussed the impact of interest rate on the values of variance swaps. We concluded that, although variance swaps are volatility derivatives speculating volatility, it is unreasonable to ignore the stochastic interest rate. This proposed analytical pricing formula can help to improve the pricing accuracy of discrete variance swaps. Our pricing approach can be extended to other stochastic interest rate and stochastic volatility models, such as the Heston–Hull–White hybrid model. However, if the correlation between the stochastic interest rate and volatility are considered, the derivation of the analytical pricing formula would become drastically more complicated. How to resolve this problem may be a potential topic in the future.

Appendix A

We now present a proof of Proposition 3.1. Applying the following transform

$$\begin{cases}\tau = T - t,\\ x = \ln S,\end{cases} \tag{A.1}$$

we can convert (3.9) to

$$\begin{cases}\dfrac{\partial U}{\partial \tau} = \dfrac{1}{2}\nu\dfrac{\partial^2 U}{\partial x^2} + \dfrac{1}{2}\sigma^2\nu\dfrac{\partial^2 U}{\partial \nu^2} + \dfrac{1}{2}\eta^2 r\dfrac{\partial^2 U}{\partial r^2} + \rho\sigma\nu\dfrac{\partial^2 U}{\partial x\partial \nu} + \left(r - \dfrac{1}{2}\nu\right)\dfrac{\partial U}{\partial x}\\ \qquad + \kappa^*(\theta^* - \nu)\dfrac{\partial U}{\partial \nu} + [\alpha^*\beta^* - (\alpha^* + B(T-\tau, T)\eta^2)r]\dfrac{\partial U}{\partial r},\\ U(x, \nu, r, 0) = H(e^x).\end{cases} \tag{A.2}$$

Performing the generalized Fourier transform to (A.2) with respect to the variable x, we obtain the following equation for $\widetilde{U}(\omega, \nu, r, \tau) = \mathcal{F}[U(x, \nu, r, \tau)]$

$$\begin{cases}\dfrac{\partial \widetilde{U}}{\partial \tau} = \dfrac{1}{2}\sigma^2\nu\dfrac{\partial^2 \widetilde{U}}{\partial \nu^2} + \dfrac{1}{2}\eta^2 r\dfrac{\partial^2 \widetilde{U}}{\partial r^2} + [\kappa^*\theta^* + (\rho\sigma\omega j - \kappa^*)\nu]\dfrac{\partial \widetilde{U}}{\partial \nu}\\ \qquad + [\alpha^*\beta^* - (\alpha^* + B(T-\tau, T)\eta^2)r]\dfrac{\partial \widetilde{U}}{\partial r} + \left[-\dfrac{1}{2}(\omega i + \omega^2)\nu + r\omega i\right]\widetilde{U},\\ \widetilde{U}(\omega, \nu, r, 0) = \mathcal{F}[H(e^x)].\end{cases} \tag{A.3}$$

Following the solution procedure of Heston's (1993), the solution to the above PDE system can be assumed to be of the following form:

$$\widetilde{U}(\omega, \nu, r, \tau) = e^{C(\omega,\tau)+D(\omega,\tau)\nu+E(\omega,\tau)r}\widetilde{U}(\omega, \nu, r, 0). \tag{A.4}$$

We can then substitute this function into (A.3) to reduce it to the following system of three ordinary differential equations:

$$\begin{cases}\dfrac{dD}{d\tau} = \dfrac{1}{2}\sigma^2 D^2 + (\rho\omega\sigma i - \kappa^*)D - \dfrac{1}{2}(\omega^2 + \omega i),\\ \dfrac{dE}{d\tau} = \dfrac{1}{2}\eta^2 E^2 - (\alpha^* + B(T-\tau, T)\eta^2)E + \omega i,\\ \dfrac{dC}{d\tau} = \kappa^*\theta^* D + \alpha^*\beta^* E,\end{cases} \tag{A.5}$$

with the initial conditions

$$C(\omega, 0) = 0, \quad D(\omega, 0) = 0, \quad E(\omega, 0) = 0. \tag{A.6}$$

Based on the ODEs, only the function D can be solved analytically as

$$D(\tau) = \frac{a+b}{\sigma^2}\frac{1 - e^{b\tau}}{1 - ge^{b\tau}}, \quad a = \kappa^* - \rho_{12}\sigma\omega i,$$

$$b = \sqrt{a^2 + \sigma^2(\omega^2 + \omega i)}, \quad g = \frac{a+b}{a-b},$$

whereas numerical integration is used to obtain the solutions of the functions E and C using standard mathematical software package, e.g., MATLAB.

Note that the Fourier transform variable ω appears as a parameter in function C, D and E. After performing the inverse Fourier transform in form, we obtain the solution to (A.2) as follows:

$$\begin{aligned}U(x, \nu, r, \tau) &= \mathcal{F}^{-1}[\widetilde{U}(\omega, \nu, r, \tau)]\\ &= \mathcal{F}^{-1}[e^{C(\omega,\tau)+D(\omega,\tau)\nu+E(\omega,\tau)r}\mathcal{F}[H(e^x)]].\end{aligned} \tag{A.7}$$

Appendix B

In this appendix, we show how to derive an expression of the expectation $\mathbb{E}_0^T[e^{\phi r(t_{j-1})}]$. The procedure also applies to the expression $\mathbb{E}_0^T[e^{\phi\nu(t_{j-1})}]$. We define

$$f(\phi, r, \tau) = \mathbb{E}_t^T[e^{\phi r(t+\tau)}]. \tag{B.1}$$

Given that the stochastic process of $r(t)$ follows the equation in (3.6) under the forward probability measure $\mathbb{Q}^T$, applying the Feynman–Kac formula, we can derive that $f(\phi, r, \tau)$ satisfies the following PDE:

$$\begin{cases} \dfrac{\partial f}{\partial \tau} = \dfrac{1}{2}\eta^2 r \dfrac{\partial^2 f}{\partial r^2} + [\alpha^*\beta^* - (\alpha^* + B(t_{j-1} - \tau, T)\eta^2)r]\dfrac{\partial f}{\partial r}, \\ f(\phi, r, \tau = 0) = e^{\phi r}, \end{cases} \tag{B.2}$$

whose solution has the form $f(\phi, r, \tau) = e^{F(\phi,\tau)+H(\phi,\tau)r}$. Substituting this function into (B.2), we obtain the following system of ordinary differential equations:

$$\begin{cases} \dfrac{dH}{d\tau} = \dfrac{1}{2}\eta^2 H^2 - (\alpha^* + B(t_{j-1} - \tau, T)\eta^2)H, \\ \dfrac{dF}{d\tau} = \alpha^*\beta^* H, \end{cases} \tag{B.3}$$

with the initial conditions

$$H(\phi, \tau) = \phi, \quad F(\phi, \tau) = 0. \tag{B.4}$$

The solution to this system is retrieved via numerical integration conducted using the MATLAB software. Next, the function $h(\phi, \nu, \tau) = \mathbb{E}_t^T[e^{\phi\nu(t+\tau)}]$ is defined in order to derive an expression of $\mathbb{E}_0^T[e^{\phi\nu(t_{j-1})}]$. The solution has the form $h(\phi, \nu, \tau) = e^{L(\phi,\tau)+M(\phi,\tau)\nu}$ with initial conditions $M(\phi, 0) = \phi$ and $L(\phi, 0) = 0$. Implementing the same procedure as that in above results, we obtain

$$\begin{cases} M(\phi, \tau) = \dfrac{\phi e^{-\kappa^*\tau}}{1 - \dfrac{\sigma^2\phi(1 - e^{-\kappa^*\tau})}{2\kappa^*}}, \\ L(\phi, \tau) = \dfrac{-2\kappa^*\theta^*}{\sigma^2} \ln\left(1 - \dfrac{\sigma^2\phi(1 - e^{-\kappa^*\tau})}{2\kappa^*}\right). \end{cases}$$

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