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SOLVING BIPOLAR FULLY FUZZY SYLVESTER MATRIX EQUATIONS WITH NEGATIVE FUZZY NUMBERS

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ABSTRACT

Sylvester matrix equations play a crucial role in control theory for controller design. Bipolar Fully Fuzzy Sylvester Matrix Equations (FFSME), incorporating both positive and negative components, are employed in controller design to address uncertainties that may affect a system's performance and stability. However, there is not much existing research on combining bipolar fuzzy numbers and FFSME, and most of them mainly deal with positive coefficients. Thus, this paper presents a method that enables solving the negative coefficient of bipolar FFSME in the form of Left-Right (LR) triangular fuzzy numbers using an Associated Bipolar Linear System (ABLS). To obtain the ABLS, bipolar FFSME is transformed into a bipolar Fully Fuzzy Linear System (FFLS) using the Kronecker product and *Vec*-operator. Subsequently, the solution is derived through the inverse method, and the equation of the ABLS is rearranged as a bipolar fuzzy matrix. Additionally, this paper provides two numerical examples to illustrate the applicability of the constructed method.

Keywords: Fully Fuzzy Sylvester Matrix Equations, Bipolar Fuzzy Numbers, Triangular Fuzzy Numbers, Negative Fuzzy Numbers.

INTRODUCTION

Bipolar analysis refers to the application of two distinct functions and information in a given system. Zhang and Zhang (2004) provided several examples of bipolar environments, such as good and bad, friend and enemy, and cooperation and competition. While bipolar is appropriate for representing two types of information in systems, sometimes bipolar crisp numbers are insufficient for dealing with uncertainty in a specific issue. Therefore, the fuzzy concept by Zadeh (1965) was employed and first introduced by Zhang (1998) in bipolar crisp numbers as bipolar fuzzy numbers. The membership functions for bipolar fuzzy numbers are [0,1] and [-1,0], referring to two different roles of the information as positive and negative parts of the information, respectively. Based on Akram et al. (2019a) and Zhang (1998), the studies are interpreted that the membership function of (0,1] indicates the element of somewhat satisfies the property. Meanwhile, the membership function of [-1,0) indicates the element of somewhat satisfies the implicit counter-property. Accordingly, bipolar fuzzy numbers can be considered an extension of classical fuzzy numbers.

Bipolar triangular fuzzy numbers are also similar to classical fuzzy numbers, which can be categorized into three forms: parametric bipolar, Left-Right (LR) bipolar, and general bipolar. These forms of bipolar fuzzy numbers can be applied in the bipolar Fuzzy Linear System (FLS), $A\tilde{X} = \tilde{B}$, and bipolar Fully FLS (FFLS), $\tilde{A}\tilde{X} = \tilde{B}$. Akram et al. (2019a) were among the first to develop classical FLS methods for solving bipolar FLS with parametric form, employing embedding techniques from the study of Asady et al. (2005) and Friedman et al. (1998). Bipolar FFLS was solved in the same paper, but a different method was used that applies the (-1,1)-cut expansion. Subsequently, Akram et al. (2019b) used the method of simultaneous equations to obtain the bipolar fuzzy solutions for both bipolar FLS and bipolar FFLS. In Akram et al. (2020), the embedding method and the bipolar fuzzy center method were employed for bipolar FLS.

Based on Dubois and Prade (2008), it is concluded that bipolar analysis has explicit benefits in many areas of information engineering, including learning, inconsistency handling, knowledge representation, question-answering systems, and especially control systems. Bipolar control is important in designing control models, specifically implemented in engineering and mathematical fields (Prajapati & Singh, 2019; Trzaskalik et al., 2019). Additionally, the Sylvester matrix equation is among the popular mathematical models that help the controllers in describing stability analysis (Asar & Amirfakhrian, 2016). The general form of the Sylvester matrix equation is denoted as $AX + XB = C$. However, researchers have improved the Sylvester matrix equation by incorporating classical fuzzy systems into the equations: Fuzzy Sylvester Matrix Equation (FSME), $A\tilde{X} + \tilde{X}B = \tilde{C}$, and FFSME, $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$.

On the other hand, Guo (2011) investigated the analytical approach of the embedding technique, which involved reducing FSME to FLS using the Kronecker product and *Vec*-operator. Malkawi et al. (2015) further improved the reduction of FSME to FLS in fully fuzzy equations such that Fully Fuzzy Sylvester Matrix Equations (FFSME) are reduced to FFLS using the Kronecker product, *Vec*-operator and an associated linear system. In a preliminary study, Daud et al. (2016) utilized the method proposed by Malkawi et al. (2015) to compare FSME to FFSME. Following that, Daud et al. (2018) extended the method to solve FFSME with arbitrary coefficients by modifying the associated linear system described in Malkawi et al. (2015), given that FFSME involves positive coefficients. Several years later, Elsayed et al. (2020) and Elsayed et al. (2022) applied a similar method as Malkawi et al. (2015) and Daud et al. (2018) to address FFSME with positive negative coefficients and extended it to handle arbitrary generalized FFSME for trapezoidal fuzzy numbers, respectively. These studies have undoubtedly made significant contributions to solving FFSME.

However, the combination of bipolar and FFSME still lacks extensive research, and existing studies have primarily focused on positive coefficients (Thape & Ahmad, 2021). Therefore, this study aims to provide an algorithm for solving bipolar FFSME with negative fuzzy numbers. The paper is organized as follows: related preliminaries on the signs and arithmetic operations of bipolar fuzzy numbers are provided in Section 2. Consequently, in Section 3, the construction of a method for solving bipolar FFSME is presented, followed by two numerical examples in Section 4. Finally, Section 5 concludes this study.

PRELIMINARIES

Definitions of signs and arithmetic operations for bipolar fuzzy numbers are given.

Definition 1. (Zhang, 1998) Let X be a nonempty set. A bipolar fuzzy set $\widetilde{M}$ in X is an object having the form

$$\widetilde{M} = \left\{\left(x, \mu_{\widetilde{M}}^{P}(x), \mu_{\widetilde{M}}^{N}(x)\right) \mid x \in X\right\}, \tag{1}$$

where $\mu_{\widetilde{M}}^{P}: X \to [0,1]$ and $\mu_{\widetilde{M}}^{N}: X \to [-1,0]$.

Definition 2. (Akram et al., 2021) A bipolar Triangular Fuzzy Number (TFN) $\widetilde{M}$ is said to be a LR bipolar TFN of the form $\widetilde{M} = \big((m^P, \alpha^P, \beta^P), (m^N, \alpha^N, \beta^N)\big)_{LR}$, where its membership function satisfies

$$\mu_{\widetilde{M}}^{P}(x) = \begin{cases} L\left(\frac{m^P - x}{\alpha^P}\right), & \text{if } x \le m^P, \alpha^P > 0, \\ \\ R\left(\frac{x - m^P}{\beta^P}\right), & \text{if } x \ge m^P, \beta^P > 0, \end{cases} \tag{2}$$

where m^P, is called the mean value, and α^P, β^P, are called the LR spreads of the positive component of $\widetilde{M}$.

While

$$\mu_{\widetilde{M}}^{N}(x) = \begin{cases} L\left(-\frac{m^N - x}{\alpha^N}\right), & \text{if } x \le m^N, \alpha^N > 0, \\ \\ R\left(-\frac{x - m^N}{\beta^N}\right), & \text{if } x \ge m^N, \beta^N > 0, \end{cases} \tag{3}$$

where m^N, is called the mean value, and α^N, β^N, are called the LR spreads of the negative component of $\widetilde{M}$.

Definition 3. (Akram et al., 2021) A bipolar TFN $\widetilde{M} = \big((m^P, \alpha^P, \beta^P), (m^N, \alpha^N, \beta^N)\big)_{LR}$ is called symmetric LR bipolar TFN if and only if $\alpha^P = \beta^P$, and $\alpha^N = \beta^N$.

Definition 4. (Akram et al., 2021) The bipolar fuzzy number $\widetilde{M} = \big((m^P, \alpha^P, \beta^P), (m^N, \alpha^N, \beta^N)\big)_{LR}$ can be classified as follows:

- $\widetilde{M}$ is a positive bipolar fuzzy number if $m^P - \alpha^P \ge 0$ and $m^N - \alpha^N \ge 0$.

- $\widetilde{M}$ is a negative bipolar fuzzy number if $\beta^P + m^P \leq 0$ and $\beta^N + m^N \leq 0$.
- $\widetilde{M}$ is a zero bipolar fuzzy number if $m^P = 0, m^N = 0, \alpha^P = 0, \alpha^P = 0, \alpha^N = 0, \beta^P = 0$, and $\beta^N = 0$.
- $\widetilde{M}$ is a near-zero bipolar fuzzy number if $m^P - \alpha^P \leq 0 \leq \beta^P + m^P$ and $m^N - \alpha^N \leq 0 \leq \beta^N + m^N$.

Definition 5. (Akram et al., 2021) Two bipolar fuzzy numbers $\widetilde{M} = \big((m^P, \alpha^P, \beta^P), (m^N, \alpha^N, \beta^N)\big)_{LR}$ and $\widetilde{N} = \big((n^P, \gamma^P, \delta^P), (n^N, \gamma^N, \delta^N)\big)_{LR}$ are equal if $m^P = n^P$, $\alpha^P = \gamma^P$, $\beta^P = \delta^P$, $m^N = n^N$, $\alpha^N = \gamma^N, \beta^N = \delta^N$.

Definition 6. (Akram et al., 2021) The arithmetic operations of two bipolar TFN $\widetilde{M} = \big((m^P, \alpha^P, \beta^P), (m^N, \alpha^N, \beta^N)\big)_{LR}$ and $\widetilde{N} = \big((n^P, \gamma^P, \delta^P), (n^N, \gamma^N, \delta^N)\big)_{LR}$ are as follows:

- Addition:

$$\widetilde{M} \oplus \widetilde{N} = \big((m^P + n^P, \alpha^P + \gamma^P, \beta^P + \delta^P), (m^N + n^N, \alpha^N + \gamma^N, \beta^N + \delta^N)\big)_{LR}. \tag{4}$$

- Opposite:

$$-\widetilde{M} = \big((-m^P, \beta^P, \alpha^P), (-m^N, \beta^N, \alpha^N)\big)_{RL}. \tag{5}$$

- Subtraction:

$$\widetilde{M} \ominus \widetilde{N} = \big((m^P - n^P, \alpha^P + \delta^P, \beta^P + \gamma^P), (m^N - n^N, \alpha^N + \delta^N, \beta^N + \gamma^N)\big)_{LR}. \tag{6}$$

- Multiplication:
- If $\widetilde{M} > 0$ and $\widetilde{N} > 0$, then

$$\widetilde{M} \otimes \widetilde{N} = \big((m^P n^P, m^P \gamma^P + n^P \alpha^P, m^P \delta^P + n^P \beta^P), (m^N n^N, m^N \gamma^N + n^N \alpha^N, m^N \delta^N + n^N \beta^N)\big)_{LR}. \tag{7}$$

- If $\widetilde{M} < 0$ and $\widetilde{N} > 0$, then

$$\widetilde{M} \otimes \widetilde{N} = \big((m^P n^P, n^P \alpha^P - m^P \delta^P, n^P \beta^P - m^P \gamma^P), (m^N n^N, n^N \alpha^N - m^N \delta^N, n^N \beta^N - m^N \gamma^N)\big)_{RL}. \tag{8}$$

- If $\widetilde{M} < 0$ and $\widetilde{N} < 0$, then

$$\widetilde{M} \otimes \widetilde{N} = \big((m^P n^P, -n^P \beta^P - m^P \delta^P, -n^P \alpha^P - m^P \gamma^P),$$
$$(m^N n^N, -n^N \beta^N - m^N \delta^N, -n^N \alpha^N - m^N \gamma^N)\big)_{RL}. \tag{9}$$

- Scalar multiplication:
  Let $\lambda \in \mathbb{R}$. Then

$$\lambda \otimes \widetilde{M} = \begin{cases} \big((\lambda m^P, \lambda\alpha^P, \lambda\beta^P), (\lambda m^N, \lambda\alpha^N, \lambda\beta^N)\big)_{LR}, & \lambda \geq 0, \\ \big((\lambda m^P, -\lambda\beta^P, -\lambda\alpha^P), (\lambda m^N, -\lambda\beta^N, -\lambda\alpha^N)\big)_{RL}, & \lambda < 0. \end{cases} \tag{10}$$

Remark 1. (Akram et al., 2021) The solution of LR bipolar TFN $\widetilde{M} = \big((m^P, \alpha^P, \beta^P), (m^N, \alpha^N, \beta^N)\big)_{LR}$ will have a strong LR bipolar fuzzy solution, if $\alpha^P, \alpha^N \geq 0$, and $\beta^P, \beta^N \geq 0$. Otherwise, the solution will be weak LR bipolar fuzzy solution.

METHOD FOR SOLVING BIPOLAR FFSME

The solution of bipolar FFSME is considered an extension of classical FFSME. In solving bipolar FFSME, an Associated Bipolar Linear System (ABLS) is constructed by establishing the Kronecker product, *Vec*-operator, and reduction of bipolar FFSME to bipolar FFLS.

Definition 7. Let $\tilde{A} \in m \times m$ and $\tilde{B} \in n \times n$ .Then, the Kronecker sum of $\tilde{A}$ and $\tilde{B}$, denoted $\tilde{A} \oplus \tilde{B}$, is $nm \times nm$ matrix $(\widetilde{U}_n \otimes \tilde{A}) + (\tilde{B} \otimes \widetilde{U}_m)$. Note that, $\tilde{A} \oplus \tilde{B} \neq \tilde{B} \oplus \tilde{A}$.

Definition 8. Let $\tilde{A} = (\tilde{a}^P, \tilde{a}^N)_{ij}$, and $\tilde{B} = (\tilde{b}^P, \tilde{b}^N)_{ij}$ be two bipolar fuzzy matrices, where $\tilde{A} \in m \times m$, and $\tilde{B} \in n \times n$, respectively. The Kronecker product of $\tilde{A}$ and $\tilde{B}$ is represented as

$$\tilde{A} \otimes \tilde{B} = \begin{bmatrix} (\tilde{a}^P, \tilde{a}^N)_{11} \otimes \tilde{B} & (\tilde{a}^P, \tilde{a}^N)_{12} \otimes \tilde{B} & \cdots & (\tilde{a}^P, \tilde{a}^N)_{1n} \otimes \tilde{B} \\ (\tilde{a}^P, \tilde{a}^N)_{21} \otimes \tilde{B} & (\tilde{a}^P, \tilde{a}^N)_{22} \otimes \tilde{B} & \cdots & (\tilde{a}^P, \tilde{a}^N)_{2n} \otimes \tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ (\tilde{a}^P, \tilde{a}^N)_{m1} \otimes \tilde{B} & (\tilde{a}^P, \tilde{a}^N)_{m2} \otimes \tilde{B} & \cdots & (\tilde{a}^P, \tilde{a}^N)_{mn} \otimes \tilde{B} \end{bmatrix}.$$

Definition 9. A *Vec*-operator generates a column vector from a bipolar fuzzy matrix $\widetilde{K}$ by stacking the column vectors of

$$\widetilde{K} = \Big[\big((m_k^P, \alpha_k^P, \beta_k^P), (m_k^N, \alpha_k^N, \beta_k^N)\big)_1, \cdots, \big((m_k^P, \alpha_k^P, \beta_k^P), (m_k^N, \alpha_k^N, \beta_k^N)\big)_n\Big],$$

as

$$Vec[\widetilde{K}] = \begin{bmatrix} \left((m_k^P, \alpha_k^P, \beta_k^P), (m_k^N, \alpha_k^N, \beta_k^N)\right)_1 \\ \vdots \\ \left((m_k^P, \alpha_k^P, \beta_k^P), (m_k^N, \alpha_k^N, \beta_k^N)\right)_n \end{bmatrix}.$$

Definition 10. The unitary bipolar fuzzy matrix is a square fuzzy matrix defined as $\widetilde{U}_n = (\tilde{u}_{ij})_{n\times n}$:

$$\tilde{u}_{ij} = \begin{cases} \left((0,0,0), (0,0,0)\right), & i \neq j, \\ \left((1,0,0), (1,0,0)\right), & i = j. \end{cases}$$

In matrix form, $\widetilde{U}_n = (\tilde{u}_{ij})_{n\times n}$ is represented as follows:

$$\widetilde{U} = \begin{bmatrix} \left((1,0,0), (1,0,0)\right) & \left((0,0,0), (0,0,0)\right) & \cdots & \left((0,0,0), (0,0,0)\right) \\ \left((0,0,0), (0,0,0)\right) & \left((1,0,0), (1,0,0)\right) & \cdots & \left((0,0,0), (0,0,0)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left((0,0,0), (0,0,0)\right) & \left((0,0,0), (0,0,0)\right) & \cdots & \left((1,0,0), (1,0,0)\right) \end{bmatrix}.$$

Theorem 1. Let $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ be a bipolar FFSME and $\tilde{S}\tilde{X} = \tilde{C}$ be a bipolar FFLS. Then, $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ and $\tilde{S}\tilde{X} = \tilde{C}$ are equivalent.

Proof. Suppose $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ is a bipolar FFSME. By applying the Kronecker product, *Vec*-operator, and unitary bipolar fuzzy matrix as stated in Definitions 8, 9, and 10, respectively, we now have

$$Vec[\tilde{A}\tilde{X} + \tilde{X}\tilde{B}] = Vec[\tilde{C}],$$

$$[(\widetilde{U}\otimes\tilde{A}) + (\tilde{B}^T\otimes\widetilde{U})]Vec[\tilde{X}] = Vec[\tilde{C}]. \tag{11}$$

Therefore, Equation (11) can be represented as bipolar FFLS,

$$\tilde{S}\tilde{X} = \tilde{C}, \tag{12}$$

where $[(\widetilde{U}\otimes\tilde{A}) + (\tilde{B}^T\otimes\widetilde{U})]$ is represented as $\tilde{S}$, $Vec[\tilde{X}]$ is represented as $\tilde{X}$, and $Vec[\tilde{C}]$ is represented as $\tilde{C}$.

Conversely, suppose $\tilde{S}\tilde{X} = \tilde{C}$ is a bipolar FFLS using *Vec*-operator as

$$Vec[\tilde{S}\tilde{X}] = Vec[\tilde{C}]. \tag{13}$$

Then, by expanding the coefficient $\tilde{S}$ as $\tilde{A}\oplus\tilde{B}$ into Equation (13), which is based on Kronecker sum as stated in Definition 7, we now have

$$Vec\left[\left(\tilde{A}\oplus\tilde{B}\right)\tilde{X}\right] = Vec\left[\tilde{C}\right]. \tag{14}$$

In addition, $\tilde{A}\oplus\tilde{B}$ is $\left(\widetilde{U}\otimes\tilde{A}\right) + \left(\tilde{B}^T\otimes\widetilde{U}\right)$ is referred to as

$$Vec\left[\tilde{A}\tilde{X}\right] = \left[\left(\widetilde{U}\otimes\tilde{A}\right)\right]Vec[\tilde{X}], \tag{15}$$

and

$$Vec[\tilde{X}\tilde{B}] = \left[\left(\tilde{B}^T\otimes\widetilde{U}\right)\right]Vec[\tilde{X}]. \tag{16}$$

Therefore, Equations (15) and (16) may be represented as bipolar FFSME given by

$$\left[\left(\widetilde{U}\otimes\tilde{A}\right) + \left(\tilde{B}^T\otimes\widetilde{U}\right)\right]Vec[\tilde{X}] = Vec\left[\tilde{C}\right],$$

$$Vec\left[\tilde{A}\tilde{X} + \tilde{X}\tilde{B}\right] = Vec\left[\tilde{C}\right], \tag{17}$$

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}.$$

Thus, the theorem is proved.

In obtaining ABLS, consider $\tilde{S} = \left((m_s^P, \alpha_s^P, \beta_s^P), (m_s^N, \alpha_s^N, \beta_s^N)\right)$ as a bipolar fuzzy number, $\tilde{X} = \left((m_x^P, \alpha_x^P, \beta_x^P), (m_x^N, \alpha_x^N, \beta_x^N)\right)$ as a bipolar fuzzy solution to be obtained, $\tilde{C} = \left((m_c^P, \alpha_c^P, \beta_c^P), (m_c^N, \alpha_c^N, \beta_c^N)\right)$ and as a bipolar fuzzy vector. Then, LR bipolar FFLS can be represented as

$$\tilde{S}\tilde{X} = \tilde{C},$$

$$\left((m_s^P, \alpha_s^P, \beta_s^P), (m_s^N, \alpha_s^N, \beta_s^N)\right) \otimes \left((m_x^P, \alpha_x^P, \beta_x^P), (m_x^N, \alpha_x^N, \beta_x^N)\right) = \tilde{C}, \tag{18}$$

$$\left((m_s^P, m_s^N), (\alpha_s^P, \alpha_s^N), (\beta_s^P, \beta_s^N)\right) \otimes \left((m_x^P, m_x^N), (\alpha_x^P, \alpha_x^N), (\beta_x^P, \beta_x^N)\right) = \tilde{C}.$$

Subsequently, using arithmetic multiplication operations as defined in Definition 6 on Equation (18), we now have

$$(m_s^P, m_s^N)\cdot(m_x^P, m_x^N) = (m_c^P, m_c^N),$$

$$(m_s^P, m_s^N)\cdot(\alpha_x^P, \alpha_x^N) + (m_x^P, m_x^N)\cdot(\alpha_s^P, \alpha_s^N) = (\alpha_c^P, \alpha_c^N), \tag{19}$$

$$(m_s^P, m_s^N)\cdot(\beta_x^P, \beta_x^N) + (m_x^P, m_x^N)\cdot(\beta_s^P, \beta_s^N) = (\beta_c^P, \beta_c^N).$$

This is followed by rearranging the bipolar linear system according to similar coefficients and adding zero terms for non-similar coefficients, which yields

$$\begin{bmatrix} (m_s^P, m_s^N) + 0 + 0 \\ (\alpha_s^P, \alpha_s^N) + (m_s^P, m_s^N) + 0 \\ (\beta_s^P, \beta_s^N) + 0 + (m_s^P, m_s^N) \end{bmatrix} \begin{bmatrix} (m_x^P, m_x^N) \\ (\alpha_x^P, \alpha_x^N) \\ (\beta_x^P, \beta_x^N) \end{bmatrix} = \begin{bmatrix} (m_c^P, m_c^N) \\ (\alpha_c^P, \alpha_c^N) \\ (\beta_c^P, \beta_c^N) \end{bmatrix}. \tag{20}$$

Accordingly, the bipolar linear system in Equation (20) can be represented in a block matrix as follows:

$$\begin{bmatrix} (m_s^P, m_s^N) & 0 & 0 \\ (\alpha_s^P, \alpha_s^N) & (m_s^P, m_s^N) & 0 \\ (\beta_s^P, \beta_s^N) & 0 & (m_s^P, m_s^N) \end{bmatrix} \begin{bmatrix} (m_x^P, m_x^N) \\ (\alpha_x^P, \alpha_x^N) \\ (\beta_x^P, \beta_x^N) \end{bmatrix} = \begin{bmatrix} (m_c^P, m_c^N) \\ (\alpha_c^P, \alpha_c^N) \\ (\beta_c^P, \beta_c^N) \end{bmatrix}, \tag{21}$$

and matrix notation as

$$SX = C. \tag{22}$$

Definition 11. Let $SX = C$ be an ABLS of bipolar FFLS, $\tilde{S}\tilde{X} = \tilde{C}$, where

$$\begin{bmatrix} (m_s^P, m_s^N) & 0 & 0 \\ (\alpha_s^P, \alpha_s^N) & (m_s^P, m_s^N) & 0 \\ (\beta_s^P, \beta_s^N) & 0 & (m_s^P, m_s^N) \end{bmatrix}, \quad X = \begin{bmatrix} (m_x^P, m_x^N) \\ (\alpha_x^P, \alpha_x^N) \\ (\beta_x^P, \beta_x^N) \end{bmatrix}, \quad C = \begin{bmatrix} (m_c^P, m_c^N) \\ (\alpha_c^P, \alpha_c^N) \\ (\beta_c^P, \beta_c^N) \end{bmatrix},$$

with (m_s^P, m_s^N) is the mean value and (α_s^P, α_s^N), (β_s^P, β_s^N) are the LR spread values of bipolar fuzzy numbers in the coefficient $\tilde{S}$. Meanwhile, (m_x^P, m_x^N), (α_x^P, α_x^N), and (β_x^P, β_x^N) are mean, left, and right values of bipolar fuzzy solution in $\tilde{X}$, respectively. Then, (m_c^P, m_c^N), (α_c^P, α_c^N), and (β_c^P, β_c^N) are mean, left, and right values of bipolar fuzzy vectors in coefficient $\tilde{C}$.

Correspondingly, details of the construction for bipolar FFSME is presented in the following steps.

Step 1. Reducing bipolar FFSME to bipolar FFLS.

Based on Theorem 1, bipolar FFSME is reduced to bipolar FFLS as

$$\tilde{S}\tilde{X} = \tilde{C}, \tag{23}$$

where the $\tilde{X}$ is the bipolar fuzzy solution to be obtained.

Step 2. Converting bipolar FFLS to a bipolar linear system using ABLS.

Let ,$\tilde{S} = \left((m_s^P, \alpha_s^P, \beta_s^P), (m_s^N, \alpha_s^N, \beta_s^N)\right), \tilde{X} = \left((m_x^P, \alpha_x^P, \beta_x^P),\right.$ $\left.n_x^N, \alpha_x^N, \beta_x^N)\right)$ and $\tilde{C} = \left((m_c^P, \alpha_c^P, \beta_c^P), (m_c^N, \alpha_c^N, \beta_c^N)\right)$ be presented as bipolar FFLS in Equation (23). Then, applying ABLS and an inverse method, the solution can be represented as

$$X = S^{-1}C. \tag{24}$$

Step 3. Rearranging the solution of ABLS as bipolar fuzzy numbers. Consequently, the solution of ABLS in Step 2 can be rearranged as

$$\tilde{X} = \left[\left((m_x^P, \alpha_x^P, \beta_x^P), (m_x^N, \alpha_x^N, \beta_x^N)\right)_{ij}\right], \tag{25}$$

where Equation (25) is considered as a bipolar fuzzy solution.

NUMERICAL EXAMPLES

The construction method of bipolar FFSME has presented two examples as follows.

Example 1. Consider bipolar FFSME, $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$, where the coefficients $\tilde{A}$ and $\tilde{B}$ is a matrix of 3×3 and 2×2, respectively.

Besides, similar values on both positive and negative components of coefficients are presented as follows:

$$\begin{bmatrix} ((-5,5,4),(-5,5,4)) & ((-4,2,3),(-4,2,3)) & ((-7,2,3),(-7,2,3)) \\ ((-8,7,2),(-8,7,2)) & ((-6,3,6),(-6,3,6)) & ((-6,2,5),(-6,2,5)) \\ ((-8,3,2),(-8,3,2)) & ((-4,3,2),(-4,3,2)) & ((-7,6,2),(-7,6,2)) \end{bmatrix} \tilde{X}$$
$$+ \tilde{X} \begin{bmatrix} ((-5,2,4),(-5,2,4)) & ((-3,2,2),(-3,2,2)) \\ ((-8,4,8),(-8,4,8)) & ((-7,3,2),(-7,3,2)) \end{bmatrix}$$
$$= \begin{bmatrix} ((191,-269,-225),(191,-269,-225)) & ((204,-231,-266),(204,-231,-266)) \\ ((201,-275,-239),(201,-275,-239)) & ((233,-264,-313),(233,-264,-313)) \\ ((200,-234,-254),(200,-234,-254)) & ((223,-201,-303),(223,-201,-303)) \end{bmatrix}, \tag{26}$$

where the solution $\tilde{X}$ can be represented as

$$\tilde{X} = \begin{bmatrix} ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{11} & ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{12} \\ ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{21} & ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{22} \\ ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{31} & ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{32} \end{bmatrix} \tag{27}$$

Step 1: Reducing bipolar FFSME to bipolar FFLS using Kronecker product and *Vec*-operator.

The positive part of bipolar fuzzy numbers, $\tilde{S}$.

$$\begin{bmatrix} (-10,7,8) & (-4,2,3) & (-7,2,3) & (-8,4,8) & (0,0,0) & (0,0,0) \\ (-8,7,2) & (-11,5,10) & (-6,2,5) & (0,0,0) & (-8,4,8) & (0,0,0) \\ (-8,3,2) & (-4,3,2) & (-12,8,6) & (0,0,0) & (0,0,0) & (-8,4,8) \\ (-3,3,2) & (0,0,0) & (0,0,0) & (-12,8,6) & (-4,2,3) & (-7,2,3) \\ (0,0,0) & (-3,3,2) & (0,0,0) & (-8,7,2) & (-13,6,8) & (-6,2,5) \\ (0,0,0) & (0,0,0) & (-3,3,2) & (-8,3,2) & (-4,3,2) & (-14,9,4) \end{bmatrix}.$$

The negative part of bipolar fuzzy numbers, $\tilde{S}$.

$$\begin{bmatrix} (-10,7,8) & (-4,2,3) & (-7,2,3) & (-8,4,8) & (0,0,0) & (0,0,0) \\ (-8,7,2) & (-11,5,10) & (-6,2,5) & (0,0,0) & (-8,4,8) & (0,0,0) \\ (-8,3,2) & (-4,3,2) & (-12,8,6) & (0,0,0) & (0,0,0) & (-8,4,8) \\ (-3,3,2) & (0,0,0) & (0,0,0) & (-12,8,6) & (-4,2,3) & (-7,2,3) \\ (0,0,0) & (-3,3,2) & (0,0,0) & (-8,7,2) & (-13,6,8) & (-6,2,5) \\ (0,0,0) & (0,0,0) & (-3,3,2) & (-8,3,2) & (-4,3,2) & (-14,9,4) \end{bmatrix}.$$

Then,

$$\tilde{X}=\begin{bmatrix}\left((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N)\right)_{11}\\\left((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N)\right)_{21}\\\left((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N)\right)_{31}\\\left((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N)\right)_{12}\\\left((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N)\right)_{22}\\\left((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N)\right)_{32}\end{bmatrix},\quad \tilde{C}=\begin{bmatrix}\left((191,-269,-225),(191,-269,-225)\right)\\\left((201,-275,-239),(201,-275,-239)\right)\\\left((200,-234,-254),(200,-234,-254)\right)\\\left((204,-231,-266),(204,-231,-266)\right)\\\left((233,-264,-313),(233,-264,-313)\right)\\\left((223,-201,-303),(223,-201,-303)\right)\end{bmatrix}.$$

Step 2: Converting bipolar FFLS to the bipolar linear system using ABLS.

Given coefficient S can be collected as

(m_S^P, m_S^N)

$$=\begin{bmatrix}((-10),(-10)) & ((-4),(-4)) & ((-7),(-7)) & ((-8),(-8)) & ((0),(0)) & ((0),(0))\\((-8),(-8)) & ((-11),(-11)) & ((-6),(-6)) & ((0),(0)) & ((-8),(-8)) & ((0),(0))\\((-8),(-8)) & ((-4),(-4)) & ((-12),(-12)) & ((0),(0)) & ((0),(0)) & ((-8),(-8))\\((-3),(-3)) & ((0),(0)) & ((0),(0)) & ((-12),(-12)) & ((-4),(-4)) & ((-7),(-7))\\((0),(0)) & ((-3),(-3)) & ((0),(0)) & ((-8),(-8)) & ((-13),(-13)) & ((-6),(-6))\\((0),(0)) & ((0),(0)) & ((-3),(-3)) & ((-8),(-8)) & ((-4),(-4)) & ((-14),(-14))\end{bmatrix},$$

(α_S^P, α_S^N)

$$=\begin{bmatrix}((7),(7)) & ((2),(2)) & ((2),(2)) & ((4),(4)) & ((0),(0)) & ((0),(0))\\((7),(7)) & ((5),(5)) & ((6),(2)) & ((0),(0)) & ((4),(4)) & ((0),(0))\\((3),(3)) & ((3),(3)) & ((8),(8)) & ((0),(0)) & ((0),(0)) & ((4),(4))\\((3),(3)) & ((0),(0)) & ((0),(0)) & ((8),(8)) & ((2),(2)) & ((2),(2))\\((0),(0)) & ((3),(3)) & ((0),(0)) & ((7),(7)) & ((6),(6)) & ((2),(2))\\((0),(0)) & ((0),(0)) & ((3),(3)) & ((3),(3)) & ((3),(3)) & ((9),(9))\end{bmatrix},$$

(β_S^P, β_S^N)

$$=\begin{bmatrix}((8),(8)) & ((3),(3)) & ((3),(3)) & ((8),(8)) & ((0),(0)) & ((0),(0))\\((2),(2)) & ((10),(10)) & ((5),(5)) & ((0),(0)) & ((8),(8)) & ((0),(0))\\((2),(2)) & ((2),(2)) & ((6),(6)) & ((0),(0)) & ((0),(0)) & ((8),(8))\\((2),(2)) & ((0),(0)) & ((0),(0)) & ((6),(6)) & ((3),(3)) & ((3),(3))\\((0),(0)) & ((2),(2)) & ((0),(0)) & ((2),(2)) & ((8),(8)) & ((5),(5))\\((0),(0)) & ((0),(0)) & ((2),(2)) & ((2),(2)) & ((2),(2)) & ((4),(4))\end{bmatrix},$$

where ABLS is defined as $SX = C$.
Then, coefficients X and C are stated as

$$X=\begin{bmatrix}(m_x^P,m_x^N)_{11}\\(m_x^P,m_x^N)_{21}\\(m_x^P,m_x^N)_{31}\\(m_x^P,m_x^N)_{12}\\(m_x^P,m_x^N)_{22}\\(m_x^P,m_x^N)_{32}\\ \\(\alpha_x^P,\alpha_x^N)_{11}\\(\alpha_x^P,\alpha_x^N)_{21}\\(\alpha_x^P,\alpha_x^N)_{31}\\(\alpha_x^P,\alpha_x^N)_{12}\\(\alpha_x^P,\alpha_x^N)_{22}\\(\alpha_x^P,\alpha_x^N)_{32}\\ \\(\beta_x^P,\beta_x^N)_{11}\\(\beta_x^P,\beta_x^N)_{21}\\(\beta_x^P,\beta_x^N)_{31}\\(\beta_x^P,\beta_x^N)_{12}\\(\beta_x^P,\beta_x^N)_{22}\\(\beta_x^P,\beta_x^N)_{32}\end{bmatrix},\quad C=\begin{bmatrix}((191),(191))\\((201),(201))\\((200),(200))\\((204),(204))\\((233),(233))\\((223),(223))\\((-225),(-225))\\((-239),(-239))\\((-254),(-254))\\((-266),(-266))\\((-313),(-313))\\((-303),(-303))\\((-269),(-269))\\((-275),(-275))\\((-234),(-234))\\((-231),(-231))\\((-264),(-264))\\((-201),(-201))\end{bmatrix}.$$

Next, the solution of bipolar linear system X is obtained using the inverse method.

$$X=\begin{bmatrix}(m_x^P,m_x^N)_{11}\\(m_x^P,m_x^N)_{21}\\(m_x^P,m_x^N)_{31}\\(m_x^P,m_x^N)_{12}\\(m_x^P,m_x^N)_{22}\\(m_x^P,m_x^N)_{32}\\ \\(\alpha_x^P,\alpha_x^N)_{11}\\(\alpha_x^P,\alpha_x^N)_{21}\\(\alpha_x^P,\alpha_x^N)_{31}\\(\alpha_x^P,\alpha_x^N)_{12}\\(\alpha_x^P,\alpha_x^N)_{22}\\(\alpha_x^P,\alpha_x^N)_{32}\\ \\(\beta_x^P,\beta_x^N)_{11}\\(\beta_x^P,\beta_x^N)_{21}\\(\beta_x^P,\beta_x^N)_{31}\\(\beta_x^P,\beta_x^N)_{12}\\(\beta_x^P,\beta_x^N)_{22}\\(\beta_x^P,\beta_x^N)_{32}\end{bmatrix}=\begin{bmatrix}\begin{pmatrix}((-5),(-5))\\((-5),(-5))\\((-7),(-7))\\((-9),(-9))\\((-8),(-8))\\((-7),(-7))\end{pmatrix}\\ \begin{pmatrix}((2),(2))\\((3),(3))\\((6),(6))\\((7),(7))\\((6),(6))\\((5),(5))\end{pmatrix}\\ \begin{pmatrix}((3),(3))\\((2),(2))\\((5),(5))\\((6),(6))\\((5),(5))\\((3),(3))\end{pmatrix}\end{bmatrix}. \tag{28}$$

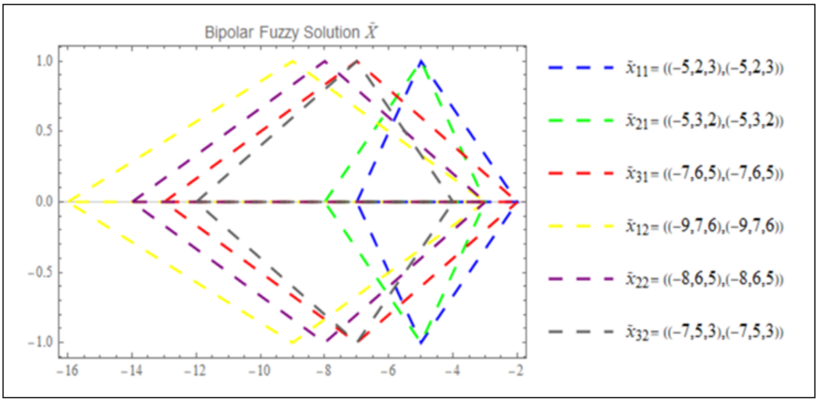
Step 3: Rearranging the solution of ABLS as a bipolar fuzzy number.

$$\tilde{X} = \begin{bmatrix} ((-5,2,3),(-5,2,3)) & ((-9,7,6),(-9,7,6)) \\ ((-5,3,2),(-5,3,2)) & ((-8,6,5),(-8,6,5)) \\ ((-7,6,5),(-7,6,5)) & ((-7,5,3),(-7,5,3)) \end{bmatrix}. \tag{29}$$

The graphical representation of Example 1 is provided in Figure 1. The bipolar fuzzy number, $\tilde{x}_{11}$ is shown in a blue dotted line, $\tilde{x}_{21}$ in a green dotted line, $\tilde{x}_{31}$ in a red dotted line, $\tilde{x}_{12}$ in a yellow dotted line, $\tilde{x}_{22}$ in a purple dotted line and $\tilde{x}_{32}$ in a grey dotted line. The values are rearranged as $\tilde{X}$ in Equation (29), and the illustration in Figure 1 represents a strong bipolar fuzzy solution based on Remark 1.

Figure 1

Bipolar Fuzzy Solution of Example 1



Verification of Solution

The solution is verified by substituting solution $\tilde{X}$ in Equation (29) into bipolar FFSME in Equation (26) with the multiplication operations in Equation (9).

$$\tilde{A}\tilde{X} = \begin{bmatrix} ((94,-114,-113),(94,-114,-113)) & ((126,-152,-169),(126,-152,-169)) \\ ((112,-141,-134),(112,-141,-134)) & ((162,-197,-223),(162,-197,-223)) \\ ((109,-101,-142),(109,-101,-142)) & ((153,-137,-208),(153,-137,-208)) \end{bmatrix}_{RL},$$

$$\tilde{X}\tilde{B} = \begin{bmatrix} ((97,-155,-112),(97,-155,-112)) & ((78,-79,-97),(78,-79,-97)) \\ ((89,-134,-105),(89,-134,-105)) & ((71,-67,-90),(71,-67,-90)) \\ ((91,-133,-112),(91,-133,-112)) & ((70,-64,-95),(70,-64,-95)) \end{bmatrix}$$

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \begin{bmatrix} ((191,-269,-225),(191,-269,-225)) & ((204,-231,-266),(204,-231,-266)) \\ ((201,-275,-239),(201,-275,-239)) & ((233,-264,-313),(233,-264,-313)) \\ ((200,-234,-254),(200,-234,-254)) & ((223,-201,-303),(223,-201,-303)) \end{bmatrix}_{RL} = \tilde{C}. \quad (30)$$

Example 2. Consider bipolar FFSME, $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$, where the coefficients $\tilde{A}$ and $\tilde{B}$ is a matrix of 3×3 and 2×2, respectively. Besides, different values on both positive and negative components of coefficients are presented as follows:

$$\begin{bmatrix} ((-9,6,4),(-5,5,4)) & ((-8,3,6),(-4,2,3)) & ((-2,2,1),(-7,2,3)) \\ ((-8,6,2),(-8,7,2)) & ((-3,2,3),(-6,3,6)) & ((-7,6,4),(-6,2,5)) \\ ((-5,5,3),(-8,3,2)) & ((-6,4,3),(-4,3,2)) & ((-9,4,8),(-7,6,2)) \end{bmatrix} \tilde{X} + \tilde{X} \begin{bmatrix} ((-8,2,9),(-5,2,4)) & ((-4,3,4),(-3,2,2)) \\ ((-5,2,4),(-8,4,8)) & ((-7,4,2),(-7,3,2)) \end{bmatrix}$$
$$= \begin{bmatrix} ((235,-253,-210),(191,-269,-225)) & ((218,-227,-237),(204,-231,-266)) \\ ((221,-248,-244),(201,-275,-239)) & ((213,-230,-256),(233,-264,-313)) \\ ((257,-373,-241),(200,-234,-254)) & ((253,-339,-252),(223,-201,-303)) \end{bmatrix}, \quad (31)$$

where the solution $\tilde{X}$ can be represented as

$$\tilde{X} = \begin{bmatrix} ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{11} & ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{12} \\ ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{21} & ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{22} \\ ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{31} & ((m_x^P,\alpha_x^P,\beta_x^P),(m_x^N,\alpha_x^N,\beta_x^N))_{32} \end{bmatrix}. \quad (32)$$

Step 1: Reducing bipolar FFSME to bipolar FFLS using Kronecker product and *Vec*-operator.

The positive part of bipolar fuzzy numbers, $\tilde{S}$.

$$\begin{bmatrix} (-17,8,13) & (-8,3,6) & (-2,2,1) & (-5,2,4) & (0,0,0) & (0,0,0) \\ (-8,6,2) & (-11,4,12) & (-7,6,4) & (0,0,0) & (-5,2,4) & (0,0,0) \\ (-5,5,3) & (-6,4,3) & (-17,6,17) & (0,0,0) & (0,0,0) & (-5,2,4) \\ (-4,3,4) & (0,0,0) & (0,0,0) & (-16,10,6) & (-8,3,6) & (-2,2,1) \\ (0,0,0) & (-4,3,4) & (0,0,0) & (-8,6,2) & (-10,6,5) & (-7,6,4) \\ (0,0,0) & (0,0,0) & (-4,3,4) & (-5,5,3) & (-6,4,3) & (-16,8,10) \end{bmatrix}.$$

Step 2: Converting bipolar FFLS to the bipolar linear system using ABLS.

Given coefficient S can be collected as

$$(m_S^P, m_S^N) = \begin{bmatrix} ((-17),(-10)) & ((-8),(-4)) & ((-2),(-7)) & ((-5),(-8)) & ((0),(0)) & ((0),(0)) \\ ((-8),(-8)) & ((-11),(-11)) & ((-7),(-6)) & ((0),(0)) & ((-5),(-8)) & ((0),(0)) \\ ((-5),(-8)) & ((-6),(-4)) & ((-17),(-12)) & ((0),(0)) & ((0),(0)) & ((-5),(-8)) \\ ((-4),(-3)) & ((0),(0)) & ((0),(0)) & ((-16),(-12)) & ((-8),(-4)) & ((-2),(-7)) \\ ((0),(0)) & ((-4),(-3)) & ((0),(0)) & ((-8),(-8)) & ((-10),(-13)) & ((-7),(-6)) \\ ((0),(0)) & ((0),(0)) & ((-4),(-3)) & ((-5),(-8)) & ((-6),(-4)) & ((-16),(-14)) \end{bmatrix},$$

$$(\alpha_S^P, \alpha_S^N) = \begin{bmatrix} ((8),(7)) & ((3),(2)) & ((2),(2)) & ((2),(4)) & ((0),(0)) & ((0),(0)) \\ ((6),(7)) & ((4),(5)) & ((6),(2)) & ((0),(0)) & ((2),(4)) & ((0),(0)) \\ ((5),(3)) & ((4),(3)) & ((6),(8)) & ((0),(0)) & ((0),(0)) & ((2),(4)) \\ ((3),(3)) & ((0),(0)) & ((0),(0)) & ((10),(8)) & ((3),(2)) & ((2),(2)) \\ ((0),(0)) & ((3),(3)) & ((0),(0)) & ((6),(7)) & ((6),(6)) & ((6),(2)) \\ ((0),(0)) & ((0),(0)) & ((3),(3)) & ((5),(3)) & ((4),(3)) & ((8),(9)) \end{bmatrix},$$

$$(\beta_S^P, \beta_S^N) = \begin{bmatrix} ((13),(8)) & ((6),(3)) & ((1),(3)) & ((4),(8)) & ((0),(0)) & ((0),(0)) \\ ((2),(2)) & ((12),(10)) & ((4),(5)) & ((0),(0)) & ((4),(8)) & ((0),(0)) \\ ((3),(2)) & ((3),(2)) & ((17),(6)) & ((0),(0)) & ((0),(0)) & ((4),(8)) \\ ((4),(2)) & ((0),(0)) & ((0),(0)) & ((6),(6)) & ((6),(3)) & ((1),(3)) \\ ((0),(0)) & ((4),(2)) & ((0),(0)) & ((2),(2)) & ((5),(8)) & ((4),(5)) \\ ((0),(0)) & ((0),(0)) & ((4),(2)) & ((3),(2)) & ((3),(2)) & ((10),(4)) \end{bmatrix},$$

where ABLS is defined as $SX = C$.

Then, coefficients X and C are stated as

$$X=\begin{bmatrix}(m_x^P,m_x^N)_{11}\\(m_x^P,m_x^N)_{21}\\(m_x^P,m_x^N)_{31}\\(m_x^P,m_x^N)_{12}\\(m_x^P,m_x^N)_{22}\\(m_x^P,m_x^N)_{32}\\ \\(\alpha_x^P,\alpha_x^N)_{11}\\(\alpha_x^P,\alpha_x^N)_{21}\\(\alpha_x^P,\alpha_x^N)_{31}\\(\alpha_x^P,\alpha_x^N)_{12}\\(\alpha_x^P,\alpha_x^N)_{22}\\(\alpha_x^P,\alpha_x^N)_{32}\\ \\(\beta_x^P,\beta_x^N)_{11}\\(\beta_x^P,\beta_x^N)_{21}\\(\beta_x^P,\beta_x^N)_{31}\\(\beta_x^P,\beta_x^N)_{12}\\(\beta_x^P,\beta_x^N)_{22}\\(\beta_x^P,\beta_x^N)_{32}\end{bmatrix},\quad C=\begin{bmatrix}((235),(191))\\((221),(201))\\((257),(200))\\((218),(204))\\((213),(233))\\((253),(223))\\((-210),(-225))\\((-244),(-239))\\((-241),(-254))\\((-237),(-266))\\((-256),(-313))\\((-252),(-303))\\((-253),(-269))\\((-248),(-275))\\((-373),(-234))\\((-227),(-231))\\((-230),(-264))\\((-339),(-201))\end{bmatrix}.$$

Next, the solution of bipolar linear system X is obtained using the inverse method.

$$X=\begin{bmatrix}(m_x^P,m_x^N)_{11}\\(m_x^P,m_x^N)_{21}\\(m_x^P,m_x^N)_{31}\\(m_x^P,m_x^N)_{12}\\(m_x^P,m_x^N)_{22}\\(m_x^P,m_x^N)_{32}\\ \\(\alpha_x^P,\alpha_x^N)_{11}\\(\alpha_x^P,\alpha_x^N)_{21}\\(\alpha_x^P,\alpha_x^N)_{31}\\(\alpha_x^P,\alpha_x^N)_{12}\\(\alpha_x^P,\alpha_x^N)_{22}\\(\alpha_x^P,\alpha_x^N)_{32}\\ \\(\beta_x^P,\beta_x^N)_{11}\\(\beta_x^P,\beta_x^N)_{21}\\(\beta_x^P,\beta_x^N)_{31}\\(\beta_x^P,\beta_x^N)_{12}\\(\beta_x^P,\beta_x^N)_{22}\\(\beta_x^P,\beta_x^N)_{32}\end{bmatrix}=\begin{bmatrix}\begin{pmatrix}((-8),(-5))\\((-6),(-5))\\((-8),(-7))\\((-7),(-9))\\((-7),(-8))\\((-9),(-7))\end{pmatrix}\\\begin{pmatrix}((3),(2))\\((3),(3))\\((4),(6))\\((3),(7))\\((5),(6))\\((2),(5))\end{pmatrix}\\\begin{pmatrix}((2),(3))\\((2),(2))\\((6),(5))\\((3),(6))\\((4),(5))\\((7),(3))\end{pmatrix}\end{bmatrix}. \quad (33)$$

Step 3: Rearranging the solution of ABLS as a bipolar fuzzy number.

$$\tilde{X} = \begin{bmatrix} ((-8,3,2),(-5,2,3)) & ((-7,3,3),(-9,7,6)) \\ ((-6,3,2),(-5,3,2)) & ((-7,5,4),(-8,6,5)) \\ ((-8,4,6),(-7,6,5)) & ((-9,2,7),(-7,5,3)) \end{bmatrix}. \quad (34)$$

The graphical representation of Example 2 is provided in Figure 2 for the positive part and Figure 3 for the negative part. The bipolar fuzzy number, $\tilde{x}_{11}$ is depicted by a blue dotted line, $\tilde{x}_{21}$ by a green dotted line, $\tilde{x}_{31}$ by a red dotted line, $\tilde{x}_{12}$ by a yellow dotted line, $\tilde{x}_{22}$ by a purple dotted line, and $\tilde{x}_{32}$ by a grey dotted line. The values are rearranged as $\tilde{X}$ in Equation (34), and the illustrations in Figures 2 and 3 represent a strong bipolar fuzzy solution based on Remark 1.

Figure 2

The Positive Part of the Bipolar Fuzzy Solution of Example 2

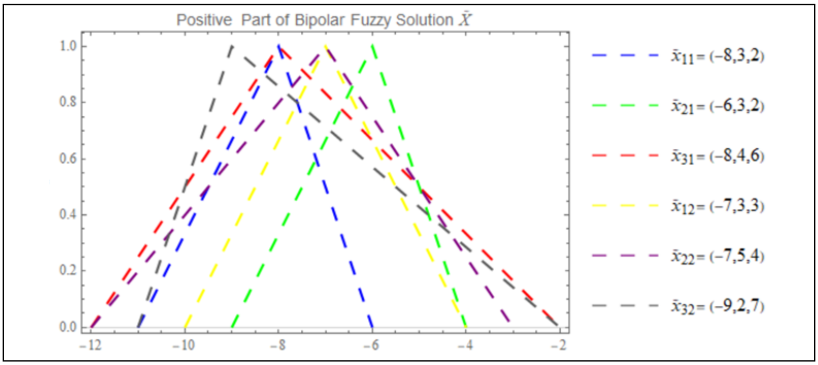
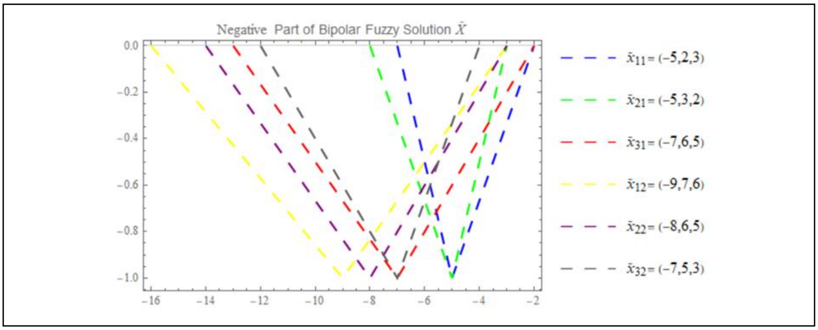


Figure 3

The Negative Part of the Bipolar Fuzzy Solution $\tilde{X}\tilde{X}$ of Example 2



Verification of Solution

The solution is verified by substituting the solution $\tilde{X}$, Equation (34) into bipolar FFSME, Equation (31) with the multiplication operations in Equation (9).

$$= \begin{bmatrix} ((136,-122,-141),(94,-114,-113)) & ((137,-152,-152),(126,-152,-169)) \\ ((138,-130,-169),(112,-141,-134)) & ((140,-156,-163),(162,-197,-223)) \\ ((148,-182,-165),(109,-101,-142)) & ((158,-216,-162),(153,-137,-208)) \end{bmatrix}_{RL},$$

$$= \begin{bmatrix} ((99,-131,-69),(97,-155,-112)) & ((81,-75,-85),(78,-79,-97)) \\ ((83,-118,-75),(89,-134,-105)) & ((73,-74,-93),(71,-67,-90)) \\ ((109,-191,-76),(91,-133,-112)) & ((95,-123,-90),(70,-64,-95)) \end{bmatrix}_{RL}.$$

$$= \begin{bmatrix} ((235,-253,-210),(191,-269,-225)) & ((218,-227,-237),(204,-231,-266)) \\ ((221,-248,-244),(201,-275,-239)) & ((213,-230,-256),(233,-264,-313)) \\ ((257,-373,-241),(200,-234,-254)) & ((253,-339,-252),(223,-201,-303)) \end{bmatrix}_{RL} = \tilde{C}^{\cdot} \tag{35}$$

CONCLUSION

This paper managed to propose a method for solving negative fuzzy numbers in bipolar FFSME, contributing to the existing body of knowledge regarding the combination of bipolar and FFSME. Bipolar FFSME with a negative number is defined by modifying the existing methods: Kronecker product and *Vec*-operator, which are used to transform bipolar FFSME to bipolar FFLS. Consequently, either a strong or weak bipolar fuzzy solution based on Remark 1 employs ABLS in bipolar FFLS and reorganizes it as a bipolar fuzzy matrix. In the future, the application of bipolar FFSME in areas such as electric circuits and complex fuzzy numbers, $a + ib$ (Ahmad et al., 2022), will be explored.

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REFERENCES

Ahmad, N., Thape, N. C. S., Daud, W. S. W., & Ibrahim, H. (2022). Solving arbitrary complex fully fuzzy linear systems for a simple electric circuit. *AIP Conference Proceedings*, *2472*(1). https://doi.org/10.1063/5.0093461

Akram, M., Ali, M., & Allahviranloo, T. (2020). Certain methods to solve bipolar fuzzy linear system of equations. *Computational and Applied Mathematics*, *39*(3), 1–28. https://doi.org/10.1007/s40314-020-01256-x

Akram, M., Allahviranloo, T., Pedrycz, W., & Ali, M. (2021). Methods for solving LR-bipolar fuzzy linear systems. *Soft Computing*, *25*(1), 85–108. https://doi.org/10.1007/s00500-020-05460-z

Akram, M., Muhammad, G., & Allahviranloo, T. (2019a). Bipolar fuzzy linear system of equations. *Computational and Applied Mathematics*, *38*(2), 1–29. https://doi.org/10.1007/s40314-019-0814-8

Akram, M., Muhammad, G., & Hussain, N. (2019b). Bipolar fuzzy system of linear equations with polynomial parametric form. *Journal of Intelligent and Fuzzy Systems*, *37*(6), 8275–8287. https://doi.org/10.3233/JIFS-190764

Asady, B., Abbasbandy, S., & Alavi, M. (2005). Fuzzy general linear systems. *Applied Mathematics and Computation*, *169*(1), 34–40. https://doi.org/10.1016/j.amc.2004.10.042

Asar, S. S., & Amirfakhrian, M. (2016). Numerical solution of Sylvester matrix equations: Application to dynamical systems. *Journal of Interpolation and Approximation in Scientific Computing*, *2016*(1), 1–13. https://doi.org/10.5899/2016/jiasc-00097

Daud, W. S. W., Ahmad, N., & Malkawi, G. (2016). A preliminary study on the solution of fully fuzzy Sylvester matrix equation. *AIP Conference Proceedings*, *1775*, 1–8. https://doi.org/10.1063/1.4965126

Daud, W. S. W., Ahmad, N., & Malkawi, G. (2018). Solving arbitrary fully fuzzy Sylvester matrix equations and its theoretical foundation. *AIP Conference Proceedings*, *2013*, 1–11. https://doi.org/10.1063/1.5054225

Dubois, D., & Prade, H. (2008). An introduction to bipolar representations of information and preference. *International Journal of Intelligent Systems*, *29*(2), 866–877. https://doi.org/10.1002/int

Elsayed, A. A. A., Ahmad, N., & Malkawi, G. (2020). On the solution of fully fuzzy Sylvester matrix equation with trapezoidal fuzzy numbers. *Computational and Applied Mathematics*, *39*(4), 1–22. https://doi.org/10.1007/s40314-020-01287-4

Elsayed, A. A. A., Ahmad, N., & Malkawi, G. (2022). Arbitrary Generalized Trapezoidal Fully Fuzzy Sylvester Matrix equation. *International Journal of Fuzzy System Applications*, *11*(1), 1–22. https://doi.org/10.4018/IJFSA.303564

Friedman, M., Ming, M., & Kandel, A. (1998). Fuzzy linear systems. *Fuzzy Sets and Systems*, *96*(2), 201–209. https://doi.org/10.1016/S0165-0114(96)00270-9

Guo, X.-B. (2011). Approximate Solution of Fuzzy Sylvester Matrix Equations. *2011 Seventh International Conference on Computational Intelligence and Security*, 52–56. https://doi.org/10.1109/CIS.2011.20

Malkawi, G., Ahmad, N., & Ibrahim, H. (2015). Solving the fully fuzzy Sylvester matrix equation with triangular fuzzy number. *Far East Journal of Mathematical Sciences*, *98*(1), 37–55. https://doi.org/10.17654/FJMSSep2015_037_055

Prajapati, M. K., & Singh, D. (2019). “ Design and analysis of speed control of bipolar stepper motor by using GA with PID approach technique ”. *International Journal of Research in Electronics and Computer Engineering*, *7*(2), 2495–2503. https://www.researchgate.net/profile/Dharmendra-Singh-63/publication/334479131_Design_and_Analysis_of_speed_control_of_Bipolar_Stepper_Motor_by_using_GA_with_PID_Approach_Technique/links/5d2d6212458515c11c337472/Design-and-Analysis-of-speed-control-of-Bipolar-Stepper-Motor-by-using-GA-with-PID-Approach-Technique.pdf

Thape, N. C. S., & Ahmad, N. (2021). Solving bipolar fully fuzzy sylvester matrix equations. *AIP Conference Proceedings*, *2365*(1), 1–12. https://doi.org/10.1063/5.0057002

Trzaskalik, T., Sitarz, S., & Dominiak, C. (2019). Bipolar method and its modifications. *Central European Journal of Operations Research*, *27*(3), 625–651. https://doi.org/10.1007/s10100-019-00615-2

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

Zhang, W. R. (1998). (Yin) (Yang) bipolar fuzzy sets. *1998 IEEE International Conference on Fuzzy Systems Proceedings -*

IEEE World Congress on Computational Intelligence, 835–840. https://doi.org/10.1109/FUZZY.1998.687599

Zhang, W. R., & Zhang, L. (2004). YinYang bipolar logic and bipolar fuzzy logic. *Information Sciences*, *165*(3–4), 265–287. https://doi.org/10.1016/j.ins.2003.05.010