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Article in International Journal of Pure and Applied Mathematics · September 2018

DOI: 10.12732/ijpam.v120i2.10

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NEAR-COMPATIBLE FACTORIZATION

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Abstract: A compatible factorization of order v , is a $v \times \frac{(v-1)}{2}$ array of distinct triples in which row i form a near-one-factor with focus i . This article aims to develop compatible factorization to display $v \times (\frac{v-1}{2} - \frac{2}{3})$ triples with minimum repetition. Through this article, we propose and define a new type of factorization called near-compatible factorization. First, we prove the existence of near-compatible factorization. Then, the construction will be presented based on difference triple method. Finally, we employ this near-compatible factorization to illustrate the development of triple design, that is called near-triad design.

AMS Subject Classification: 05B05, 05B07, 05B10, 05B40, 05C70

Key Words: near-one-factorization, λ -fold triple system, difference triple, compatible factorization, near-compatible factorization

1. Introduction

We shall review standard notations and some definitions on graph theory. K_v will denote the complete graph of order v .

A one-factor in a graph G is a set of edges in which every vertex appears exactly once. A one-factorization (briefly *OF*) of a graph G is a partition of the edge set into edge-disjoint one-factors. Obviously, the necessary condition

Received: November 30, 2017

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Revised: February 6, 2018

url: www.acadpubl.eu

Published: September 30, 2018

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in order for a graph to have an *OF* is that a graph with a one-factor must have an even number of vertices. A comprehensive background on one-factorization and related subjects can be found in [11].

A near-one-factorization (briefly *NOF*) is the closest thing to a one-factor of a set on $n - 1$ edges which cover all but one vertex. A near-on-factorization is a set of near-one-factors which covers every edge exactly once [11]. It is easy to construct an *NOF* from an *OF* by removing a common vertex. And vice versa, we can also construct *OF* from *NOF* by adjoining to each near-one-factor a new vertex. For example, when $v = 5$ we can construct *OF* from *NOF* and *NOF* from *OF* as follows:

<i>NOF</i>					<i>OF</i>				
1	2	5	3	4		0	1	2	5
2	1	4	3	5	\implies <i>OF</i>				
3	1	2	4	5	<i>NOF</i> \Leftarrow				
4	1	5	2	3		0	2	1	4
5	1	3	2	4		0	3	1	2
						0	4	1	5
						0	5	1	3
								2	4

If a near-one-factor written as:

$$F : a \ cd \ ef \ \dots \ yz,$$

this refers to

$$acd \ aef \ \dots \ ayz,$$

as the set of triples associated with F . For more, readers can refer to [3] and [6].

A balanced incomplete block design BIBD is a pair (V, T) , where V is a finite set of v objects and T is a collection of k -subsets of V called blocks, $2 \leq k < v$, such that each pair of distinct objects of V is contained in exactly λ blocks of T . The design is often described as a (v, k, λ) BIBD.

A triple system of index λ (or a λ -fold triple system), denoted by $TS(v, \lambda)$, is a BIBD with $k = 3$. On other words, we can say that a λ -fold triple system is a decomposition for λK_v , the graph with v vertices in which every two vertices are joined by λ parallel edges, into edge disjoint triangles. A triple system $TS(v, \lambda)$ with $V = Z_v$ is cyclic if, for each triple $\{c_0, c_1, c_2\} \in T$ we also have $\{c_0 + 1, c_1 + 1, c_2 + 1\} \in T$. It is simple if its triples are all distinct.

The development of triple design construction is one of the most prominent areas of research in combinatorics. Recently, studies

have been carried out in the construction of triple designs. Ibrahim and Wallis constructed a new type of factorization called compatible factorization, and they used this factorization in building up the triad design which concerned with the arrangement of distinct triples into rows satisfying certain specified conditions [6]. They proved that the existence of triad design only for $v \equiv 1$ or $5 \pmod{6}$. New algorithms have been developed for triad design for cases $v \equiv 1$ or $5 \pmod{6}$ in [5] and [8].

In the field of constructing simple cyclic designs, some decompositions of triples of Z_v into cyclic triple systems have been proposed by Tian and Wei [9] and [10]. Furthermore, they defined a large set of cyclic triple systems to be a decomposition of triples of Z_v into indecomposable cyclic designs.

In this article, some preliminaries and definitions are provided in Section 2. Then, Section 3 presents the basic concepts and construction for near-compatible factorization. The development of triple system that is called near-triad design will be discussed in Section 4. Furthermore, we use near-compatible factorization to illustrate the case near-triad design of order 9. Finally, Section 5 discusses the conclusions and future work.

2 Preliminaries and Definitions

In this section, we provide definitions and preliminaries that are needed in the sequel.

Theorem 1. [2] *In a (v, k, λ) -BIBD with b blocks each object occurs in r blocks where*

- (i) $\lambda(v - 1) = r(k - 1)$;
- (ii) $bk = vr$.

Definition 2. [3] A compatible factorization of order v , denoted by $CF(v)$, is a $v \times \frac{v-1}{2}$ array which satisfies the following conditions:

- (i) The entries in row i form a near-one-factor with focus i ;
- (ii)

The triples associated with the rows contain no repetitions.

It is evident that the necessary condition for the existence of a $CF(v)$ is that v must be odd.

Theorem 3. [3] *A compatible factorization of order v exists for every odd order $v > 3$.*

Proof. Suppose $v = 2s + 1 > 3$. The near-one-factor form the patterned starter, with i -th factor

$$i \quad (i+1)(i-1) \quad (i+2)(i-2) \dots (i+n)(i-n) \pmod{v}$$

is a compatible factorization. \square

Example 4. To construct $CF(9)$ satisfying the conditions of Definition 2, we must have 36 distinct triples are associated in 9 rows with 4 columns and one column of isolated vertex as shown in Table 1.

C_1	C_2	C_3	C_4	C_5
1	2 3	4 5	6 7	8 9
2	1 4	3 5	6 8	7 9
3	1 5	2 4	6 7	8 9
4	1 3	2 5	6 9	7 8
5	1 2	3 4	6 7	8 9
6	1 2	3 5	4 8	7 9
7	1 3	2 6	4 5	8 9
8	1 5	2 7	3 6	4 9
9	1 4	2 3	5 7	6 8

Table 1: Example of $CF(9)$.

By appending C_1 to each other columns C_2 , C_3 , C_4 , and C_5 , we obtain four triples in each row.

Definition 5. [4] A triad design on v objects, denoted by $TD(v)$, is a way of arranging $\binom{v}{3}$ distinct triples into v rows such that:

- (i) Row i contains $\frac{v-1}{2}$ triples, among which object i meets every other object precisely once, and contains also some other triples among which every object other than i occurs equally often;

- (ii) No two objects occur together twice or more in the same row;
- (iii) Each triple appears precisely once in the design.

Theorem 6. [4] *In any triad design, $v \equiv 1$ or $5 \pmod{6}$.*

Proof. Suppose there is a triad design on v objects. Then the $\binom{v}{3}$ distinct triples on v objects must be partitioned into v sets, each set containing the same number of triples. So v divides $\binom{v}{3}$. Therefore 6 divides $(v-1)(v-2)$. This implies $v \equiv 1$ or $2 \pmod{3}$. Now, v must be odd in order for the required *CF* to exist (from Theorem 3), so $v \equiv 1$ or $5 \pmod{6}$. \square

Definition 7. [7] A covering of a graph G with triangles is a triple (V, T, P) , where V is a vertex set of G , P is a subset of the edge set of λG based on V (λG is the multigraph in which every two vertices are joined by λ parallel edges), and T is a collection of triangles which partitions the union of P and the edge set of G . P is called the padding and the number $|V|$ the order of the covering (V, T, P) . If $|P|$ is as small as possible the covering is called a minimum covering with triangles (*MCT*).

3 Algorithm for Near-Compatible Factorization

In this section, we define concept of difference triple as a base towards constructing a new type of factorization called near-compatible factorization which aims to arrange $\frac{v(v-1)}{2} - \frac{2v}{3}$ triples in v rows by using near-one-factor with minimum repeated triples. This type of factorization will be employed to construct some triple designs for arranging $\binom{v}{3}$ triples into v rows for $v \equiv 3 \pmod{6}$ with minimum repetitions of triples. Then, a construction for near-compatible factorization will be presented.

For any edge $\{x, y\}$ in K_v with $V(K_v) = Z_v$, we define the difference of the edge $\{x, y\}$ by $d = \min\{|y - x|, v - |y - x|\}$. For $a_i \in Z_v - \{0\}$ and $1 \leq a_i \leq \lfloor v/2 \rfloor$, $i = 1, 2, 3$, if $a_1 + a_2 + a_3 \equiv 0 \pmod{v}$ or $a_1 + a_2 \equiv a_3 \pmod{v}$, then (a_1, a_2, a_3) is called a difference triple. The orbit of triples corresponding to a difference triple (a_1, a_2, a_3) is $\{1 + i, (1 + i) + a_1, (1 + i + a_1) + a_2 : i \in Z_v\}$, and the triple $\{1, 1 + a_1, 1 + a_1 + a_2\}$ is called a starter triple.

For $v \equiv 3 \pmod{6}$, we partition the difference triples of Z_v into three types:

Type 1: $(a, a, 2a)$, where $1 \leq a \leq \frac{v-1}{2}$. In particular, the following difference triple $(\frac{v}{3}, \frac{v}{3}, \frac{v}{3})$ has a short orbit of size $\frac{v}{3}$. The number of difference triples for this type is $\frac{v-1}{2}$.

Type 2: (a, b, c) , where $a < b < c$. The number of this type of difference triples is $\frac{(v-3)^2}{12}$.

Type 3: (a, c, b) the adjoined difference triples of Type 2, and they have the same number of difference triples.

For example, consider $v = 9$. Then the set of all distinct triples on 9 objects equals $\binom{9}{3}$ triples can be arranged by the difference triples as shown in Table 2.

Type 1 of difference triples. Type 2 and Type 3 of difference triples.

(1,1,2)	(2,2,4)	(3,3,3)	(4,4,1)	(1,2,3)	(1,3,2)	(2,3,4)	(2,4,3)	(1,3,4)	(1,4,3)
1 2 9	1 3 8	1 4 7	1 5 6	1 2 4	1 2 8	1 3 6	1 3 7	1 2 5	1 2 7
2 3 1	2 4 9	2 5 8	2 6 7	2 3 5	2 3 9	2 4 7	2 4 8	2 3 6	2 3 8
3 4 2	3 5 1	3 6 9	3 7 8	3 4 6	3 4 1	3 5 8	3 5 9	3 4 7	3 4 9
4 5 3	4 6 2		4 8 9	4 5 7	4 5 2	4 6 9	4 6 1	4 5 8	4 5 1
5 6 4	5 7 3		5 9 1	5 6 8	5 6 3	5 7 1	5 7 2	5 6 9	5 6 2
6 7 5	6 8 4		6 1 2	6 7 9	6 7 4	6 8 2	6 8 3	6 7 1	6 7 3
7 8 6	7 9 5		7 2 3	7 8 1	7 8 5	7 9 3	7 9 4	7 8 2	7 8 4
8 9 7	8 1 6		8 3 4	8 9 2	8 9 6	8 1 4	8 1 5	8 9 3	8 9 5
9 1 8	9 2 7		9 4 5	9 1 3	9 1 7	9 2 5	9 2 6	9 1 4	9 1 6

Table 2: The set all triples on 9 objects.

When $v = 9$ we have $\binom{9}{3} = 84$ different triples. We used 36 different triples in the construction of $CF(9)$ as shown in Table 1, and we have 48 different triples have been left. Since 48 not divisible by 9, hence, we can't arrange the remaining triples to construct $TD(9)$. A natural question to ask then, how "close" can we arrange $\binom{9}{3}$ triples into 9 rows.

Even though the triad design is a completion to compatible factorization and compatible factorization exists for every odd order more than 3, then triad design does not exist for $v \equiv 3 \pmod{6}$ because of $\binom{v}{3}$ is not divisible by v . For this purpose, we aim to develop the compatible factorization in order to construct a new type of factorization will be called near-compatible factorization can

be contributed to arranging $\binom{v}{3}$ triples into v rows for $v \equiv 3 \pmod{6}$ with minimum repetitions of triples.

Definition 8. A near-compatible factorization on v objects, denoted by $NCF(v)$, is a $v \times \frac{v-1}{2}$ array that satisfies the following conditions:

- (i) The entries in row i form a near-one-factor with focus i ;
- (ii) The triples associated with the rows contain minimum repeated triples.

Lemma 9. *There exists a near-compatible factorization for $v \equiv 3 \pmod{6}$.*

Proof. Suppose $v = 6n + 3$. Then v is odd and divisible by 3. Since v is divisible by 3, then there exists a difference triple $(\frac{v}{3}, \frac{v}{3}, \frac{v}{3})$ which represents a one-third column (a short orbit of size $\frac{v}{3}$) and this column contains the minimum repeated triples. Since v is odd, then near-one-factor exists. So there exists a near-compatible factorization of order v . \square

Theorem 10. *There are exactly $\frac{v}{3}$ triples repeated thrice in near-compatible factorization of order $v \equiv 3 \pmod{6}$.*

Proof. Suppose $v \equiv 3 \pmod{6}$. Let (V, T) be a $TS(v, 3)$, then $V = \{1, 2, 3, \dots, v\}$. Let T^* be a collection of all triples corresponding to Type 1 of difference triple, and let T^{**} be a multiset of triples corresponding to a difference triple $(\frac{v}{3}, \frac{v}{3}, \frac{v}{3})$ repeated twice.

By applying a minimum covering with 3-fold triple system (MCT) on Type 1 of difference triples, we get $(V, T^* \cup T^{**}, P)$ is a minimum covering of order v with 3-fold triple system, where P is a multi-edge set which partitions the multiset $E(T^{**})$ of edges of T^{**} . Now T^{**} contains $\frac{v}{3}$ triples repeated twice, so a near-compatible factorization of order v which formed by $T^* \cup T^{**}$ contains $\frac{v}{3}$ triples repeated thrice. \square

Now by choosing the orbits of triples corresponding to Type 1 of difference triples and repeating the triples of short orbit twice such that the number of repeated triples minimized, we get a cyclic triple system of index $\lambda = 3$ called near-compatible factorization as shown for case $v = 9$ in Table 3.

(1,2,1)	(2,4,2)	(3,3,3)	(4,1,4)
1 2 9	1 3 8	1 4 7	1 5 6
2 3 1	2 4 9	2 5 8	2 6 7
3 4 2	3 5 1	3 6 9	3 7 8
4 5 3	4 6 2	4 7 1	4 8 9
5 6 4	5 7 3	5 8 2	5 9 1
6 7 5	6 8 4	6 9 3	6 1 2
7 8 6	7 9 5	7 1 4	7 2 3
8 9 7	8 1 6	8 2 5	8 3 4
9 1 8	9 2 7	9 3 6	9 4 5

Table 3: Desired near-compatible factorization of order 9.

In this case, we have three repeated triples and each row contains one of them. Note that the triples are unordered. For example $\{1 4 7\}$, $\{4 7 1\}$ are considered the same triple.

The next steps are construction for near-compatible factorization. We exemplify the construction for two cases $v = 9$ and $v = 15$ as a groundwork in developing our near-compatible factorization.

Step 1: Suppose there are $6n + 3$ vertices. Divide the vertices as follows: the even vertices on the left beginning from $2, 4, \dots, 6n + 2$ and the odd vertices on the right beginning from $3, 5, \dots, 6n + 3$. And vertex 1 will be put on the top middle of the partition as shown in Figure 1.

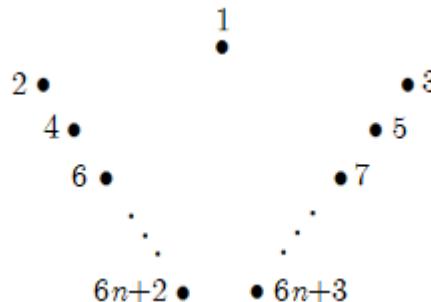
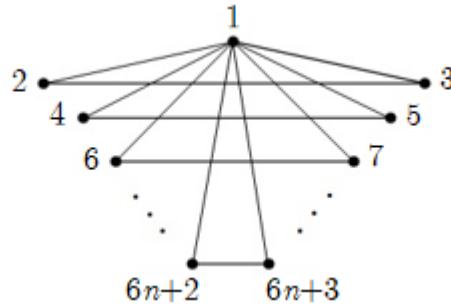
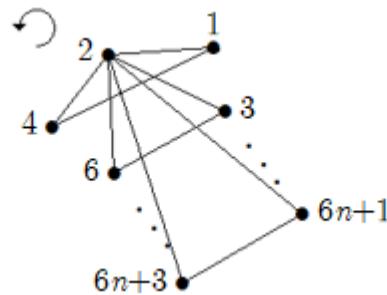


Figure 1: Partition of vertices.

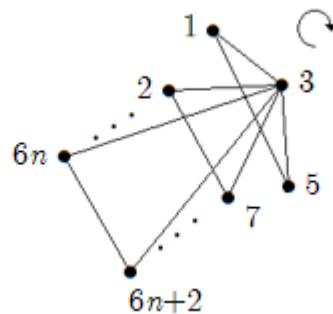
Step 2: To form the triples (triangles) of the first factor F_1 , consider vertex 1 be the reference vertex. Then connect each even and odd vertex with vertex 1 as shown in Figure 2.

Figure 2: The first factor of $NCF(v)$.

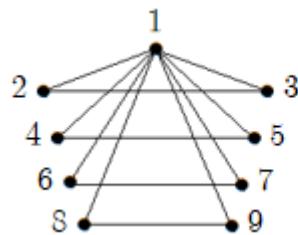
Step 3: To form the even factors, for example F_2 , consider vertex 2 be the reference vertex and rotate all the triangles counter-clockwise as shown in Figure 3. Continue in this pattern to create the remaining even factors $F_4, F_6, \dots, F_{6n+2}$.

Figure 3: The second factor of $NCF(v)$.

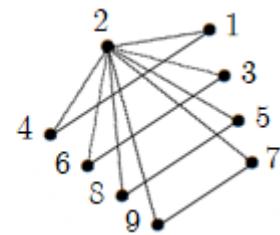
Step 4: To form the odd factors, for example F_3 , consider vertex 3 be the reference vertex and rotate all the triangles clockwise as shown in Figure 4. Continue in this pattern to form the remaining odd factors $F_5, F_7, \dots, F_{6n+3}$.

Figure 4: The third factor of $NCF(v)$.

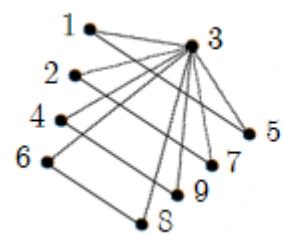
Example 11. The construction of $NCF(9)$.



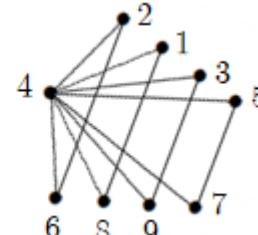
$F_1 : 1 \quad 23 \quad 45 \quad 67 \quad 89$



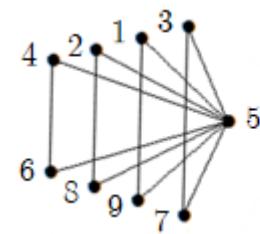
$F_2 : 2 \quad 14 \quad 36 \quad 58 \quad 79$



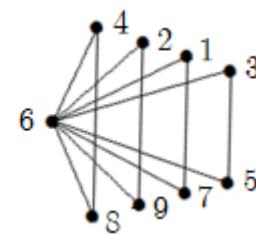
$F_3 : 3 \quad 15 \quad 27 \quad 49 \quad 68$



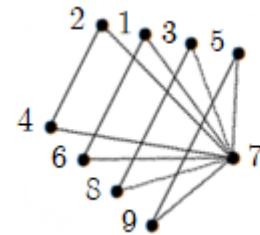
$F_4 : 4 \quad 26 \quad 18 \quad 39 \quad 57$



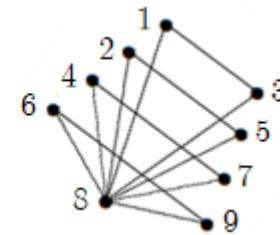
$F_5 : 5 \quad 37 \quad 19 \quad 28 \quad 46$



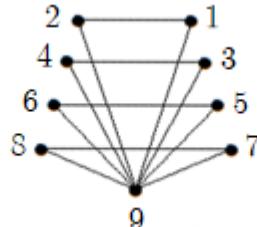
$F_6 : 6 \quad 48 \quad 29 \quad 17 \quad 35$



$F_7 : 7 \quad 59 \quad 38 \quad 16 \quad 24$



$F_8 : 8 \quad 69 \quad 47 \quad 25 \quad 13$

Figure 5: Construction of $NCF(9)$.

From the construction of $NCF(9)$ above we have the following table.

$F_1 :$	1	2	3	4	5	6	7	8	9
$F_2 :$	2	1	4	3	6	5	8	7	9
$F_3 :$	3	1	5	2	7	4	9	6	8
$F_4 :$	4	2	6	1	8	3	9	5	7
$F_5 :$	5	3	7	1	9	2	8	4	6
$F_6 :$	6	4	8	2	9	1	7	3	5
$F_7 :$	7	5	9	3	8	1	6	2	4
$F_8 :$	8	6	9	4	7	2	5	1	3
$F_9 :$	9	7	8	5	6	3	4	1	2

Table 4: Example of $NCF(9)$.

Note that both compatible factorization $CF(v)$ and near-compatible factorization $NCF(v)$ are near-one-factorization. $CF(v)$ contains no repetitions of triples, while $NCF(v)$ contains only $\frac{v}{3}$ triples repeated thrice.

Example 12. Given $v = 15$ and the objects are labeled as 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e and f. Then the following table provides an example of $NCF(15)$ based on the above construction steps.

$F_1 :$	1	2 3	4 5	6 7	8 9	a b	c d	e f
$F_2 :$	2	1 4	3 6	5 8	7 a	9 c	b e	d f
$F_3 :$	3	1 5	2 7	4 9	6 b	8 d	a f	c e
$F_4 :$	4	2 6	1 8	3 a	5 c	7 e	9 f	b d
$F_5 :$	5	3 7	1 9	2 b	4 d	6 f	8 e	a c
$F_6 :$	6	4 8	2 a	1 c	3 e	5 f	7 d	9 b
$F_7 :$	7	5 9	3 b	1 d	2 f	4 e	6 c	8 a
$F_8 :$	8	6 a	4 c	2 e	1 f	3 d	5 b	7 9
$F_9 :$	9	7 b	5 d	3 f	1 e	2 c	4 a	6 8
$F_a :$	a	8 c	6 e	4 f	2 d	1 b	3 9	5 7
$F_b :$	b	9 d	7 f	5 e	3 c	1 a	2 8	4 6
$F_c :$	c	a e	8 f	6 d	4 b	2 9	1 7	3 5
$F_d :$	d	b f	9 e	7 c	5 a	3 8	1 6	2 4
$F_e :$	e	c f	a d	8 b	6 9	4 7	2 5	1 3
$F_f :$	f	d e	b c	9 a	7 8	5 6	3 4	1 2

Table 5: Example of $NCF(15)$.

In this case, we have five triples repeated thrice, and each row forms a near-one-factor.

4 Towards Near-Triad Design

This section explains how near-compatible factorization can be contributed to solving the problem of arranging $\binom{v}{3}$ triples into v rows for $v \equiv 3 \pmod{6}$ with minimum repetitions of triples, we define a new triple design will be called near-triad design.

Definition 13. A near-triad design of order v , denoted by $NTD(v)$, is a way of arranging $\binom{v}{3}$ triples into v rows such that:

- (i) Row i contains $\frac{v-1}{2}$ triples, among which object i meets every other objects precisely once, and contains also some other triples among which the objects other than i occurs equally often;
- (ii) Each triple appears once except $\frac{v(v-1)}{6}$ triples appear exactly thrice in the design.

Theorem 14. For each $v \equiv 3 \pmod{6}$, there exists a near-triad design of order v .

Proof. Suppose $v = 6n + 3$. There are $\binom{v}{3}$ distinct triples on v objects. By adding $\frac{v(v-1)}{6}$ triples twice to $\binom{v}{3}$ distinct triples (the set of all triples on v objects), we get a multiset of $\binom{v}{3} + (2)\frac{v(v-1)}{6} = \frac{(v-1)v^2}{6}$ triples among which only $\frac{v(v-1)}{6}$ triples repeated thrice.

Since v divides $\frac{(v-1)v^2}{6}$, then a multiset of $\frac{(v-1)v^2}{6}$ triples can be partitioned into v multi-subsets of the same cardinality, each multi-subset containing $\frac{(v-1)v}{6}$ triples. So any triple design of v rows formed by a multiset of $\frac{(v-1)v^2}{6}$ triples must contains $\frac{(v-1)v}{6}$ columns. Since $v = 6n + 3$, then there exists a $NCF(v)$ (from Lemma 9). By subtracting $\frac{(v-1)}{2}$ (the number of triples in row i of $NCF(v)$) from $\frac{(v-1)v}{6}$ (the number of triples in row i of the design) we get:

$$\frac{(v-1)v}{6} - \frac{(v-1)}{2} = \frac{(v-1)(v-3)}{6}$$

This implies that $(v-1)$ divides $\frac{(v-1)v}{6} - \frac{(v-1)}{2}$. Therefore the objects other than i can occur equally in the remaining triplets of row i . So there exists a $NTD(v)$. \square

Lemma 15. *A set of all triples on Z_v is a $TS(v, v-2)$.*

Proof. A set of all triples on Z_v is $(v, 3, \lambda)$ -BIBD contains $\binom{v}{3}$ distinct triples. By substituting $b = \binom{v}{3}$ and $k = 3$ in (i) of Theorem 1, we have $r = \frac{(v-1)(v-2)}{2}$. Then by substituting $r = \frac{(v-1)(v-2)}{2}$ in (ii) of Theorem 1, we obtain $\lambda = (v-2)$. So a set of all triples on Z_v is a $TS(v, v-2)$. \square

Theorem 16. *For $v \equiv 3 \pmod{6}$, a near-triad design of order v is a v -fold triple system contains a minimum of repeated triples.*

Proof. Suppose $v = 6n + 3$. Since a set of $\binom{v}{3}$ distinct triples is a $TS(v, v-2)$ (from Lemma 15), and $NTD(v)$ contains repeated triples then $NTD(v)$ is a $(v, 3, \lambda)$ -BIBD with $\lambda > v-2$.

By substituting $\lambda = (v-1)$ in (ii) of Theorem 1, we get $r = \frac{(v-1)^2}{2}$. Then by substituting $r = \frac{(v-1)^2}{2}$ in (i) of Theorem 1, we obtain $b = \frac{v(v-1)^2}{6}$. Since $v = 6n + 3$ then $b = 2(2n+1)(3n+1)^2$ is not divisible by $6n + 3$. So the triples of $(v, 3, v-1)$ -BIBD cant be arranged into v rows. Therefore $(v, 3, v-1)$ -BIBD is not near-triad design.

By substituting $\lambda = v$ in (ii) of Theorem 1, we get $r = \frac{v(v-1)}{2}$. Then by substituting $r = \frac{v(v-1)}{2}$ in (i) of Theorem 1, we obtain $b = \frac{v^2(v-1)}{6}$. Since $v = 6n + 3$ then $b = (6n + 3)(3n + 1)(2n + 1)$ is divisible by $6n + 3$. So the triples of $(v, 3, v)$ -BIBD can be arranged into v rows. Therefore $(v, 3, v)$ -BIBD is a near-triad design of order v with least value of $\lambda > v - 2$. Now a near-triad design of order v is a v -fold triple system contains a minimum of repeated triples. \square

The following table provides an example of $NTD(9)$ to illustrate how near-compatible factorization can be used in the construction of near-triad design.

(1,2,1)	(2,4,2)	(3,3,3)	(1,4,4)	(1,2,3)	(1,3,2)	(2,3,4)	(2,4,3)	(1,3,4)	(1,4,3)		
1 2 9	1 3 8	1 4 7	1 5 6	3 4 6	2 4 5	2 6 8	7 9 4	8 9 3	5 8 9	5 7 3	2 6 7
2 3 1	2 4 9	2 5 8	2 6 7	5 6 8	4 6 7	6 1 3	8 1 5	9 1 4	9 3 4	8 9 7	5 7 3
3 4 2	3 5 1	3 6 9	3 7 8	4 5 7	6 8 9	4 8 1	9 2 6	1 2 5	1 4 5	2 6 7	8 9 7
4 5 3	4 6 2	4 7 1	4 8 9	6 7 9	5 7 8	5 9 2	1 3 7	2 3 6	8 2 3	8 1 6	5 9 1
5 6 4	5 7 3	5 8 2	5 9 1	8 9 2	7 9 1	9 4 6	2 4 8	3 4 7	3 6 7	2 3 1	8 1 6
6 7 5	6 8 4	6 9 3	6 1 2	7 8 1	9 2 3	7 2 4	3 5 9	4 5 8	4 7 8	5 9 1	2 3 1
7 8 6	7 9 5	7 1 4	7 2 3	9 1 3	8 1 2	8 3 5	4 6 1	5 6 9	2 5 6	2 4 9	8 3 4
8 9 7	8 1 6	8 2 5	8 3 4	2 3 5	1 3 4	3 7 9	5 7 2	6 7 1	6 9 1	5 6 4	2 4 9
9 1 8	9 2 7	9 3 6	9 4 5	1 2 4	3 5 6	1 5 7	6 8 3	7 8 2	7 1 2	8 3 4	5 6 4

Table 6: Example of $NTD(9)$.

The first four columns which have Type 1 of difference triples form $NCF(9)$. In this case, we have 12 triples repeated thrice in $NTD(9)$ among which 3 triples repeated thrice in $NCF(9)$.

5 Conclusions

In this article, we have investigated new triple designs with minimum repetitions of triples for the odd case $v \equiv 3 \pmod{6}$. Especially, we have defined and proved the existence of $NCF(v)$. We have also constructed our near-compatible factorization, and this construction has been exemplified for two cases $v = 9$ and $v = 15$. Then we have defined $NTD(v)$ as an application for $NCF(v)$. We expect this factorization can be developed and extended to construct new triple designs with minimum repetitions of triples for even cases $v \equiv 0, 2, 4 \pmod{6}$.

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