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SOLUTION OF LR-FUZZY LINEAR SYSTEMWITH TRAPEZOIDAL FUZZY NUMBERUSING MATRIX THEORY

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ABSTRACT

This study provides solutions to aLR-fuzzy linear system (LR-FLS) with trapezoidal fuzzy number using matrix theory. The components of the LR-FLS are represented in block matrices and vectors to produce an equivalent linear system. Then, the solution can be obtained using any classical linear system, such as an inversion matrix. In this method, fuzzy operations are not required and the solution obtained is either fuzzy or non-fuzzy exact solution. Finally, several examples are given to illustrate the ability of the proposed method.

Keywords: fuzzy linear system, trapezoidal fuzzy number, block matrix.

INTRODUCTION

A linear system of equations is considered the simplest model in solving mathematical problems. However, the coefficients of these systems are usually not completely known. Therefore, fuzzy theory (Zadeh, 1965) has been employed to the linear system and called a fuzzy linear system (FLS), in which the left-hand side is a real number and the right-hand side is a fuzzy vector.

Friedman *et al.* (1998) introduced a generic model for solving an FLS with fuzzy numbers in a particular form. In his model, fuzzy solutions are classified into strong or weak fuzzy solutions to the original fuzzy system. Allahviranloo *et al.* (2011) revealed through a counterexample that a weak fuzzy solution does not always yield a fuzzy number vector. Besides that, Left-right FLS (LR-FLS) which is an extension of FLS was used, where the fuzzy numbers are triangular fuzzy numbers. Several methods to solve LR-FLS were developed such as Ghanbari and Mahdavi-Amiri (2010), Allahviranloo *et al.* (2012a)

Meanwhile, Nasseri *et al.* (2011) solved the LR-FLS with trapezoidal fuzzy numbers. He proposed a numerical method based on trapezoidal fuzzy numbers for associated triangular fuzzy numbers and a particular form of fuzzy numbers. Later, Allahviranloo *et al.* (2012b, c) introduced metric functions to provide the nearest solution to LR-FLS. The LR-FLS was transformed into the minimization problem, where the constraints of non-linear programing guarantee that the solution is an LR fuzzy number. Allahviranloo *et al.* (2012b) used a similar technique in Allahviranloo *et al.* (2012c) to solve the symmetric LR-FLS and found the nearest approximate symmetric fuzzy solution to a symmetric LR -FLS, where the exact solution is a non-fuzzy solution. However, the solution of those existing methods led to more computation time which depend heavily on fuzzy operation, ranking function, linear or nonlinear programming (Malkawi *et al*., 2014a). Moreover, the size of fuzzy systems are restricted to$n = 2$ or 3(Nasseri *et al.* 2011; Allahviranloo *et al*., 2012b, c).

In this study, we solve the LR-FLS with trapezoidal fuzzy numbers using matrix theory to overcome the aforementioned problem. The coefficients on the left-hand side are classified into positive and negative matrices and represented in block matrix, and the LR fuzzy vector on the right-hand side is represented by one crisp vector, then LR-FLS is transferred to an equivalent linear system, which allows the system uses the classical method as the matrix inversion method.

This paper consists of 4 sections. In Section 2, the basic definitions of fuzzy set theory are reviewed. The construction of the new method is discussed in Section 3. Finally, the conclusion is given in Section 4.

PRELIMINARIES

In this section, basic definitions and notions of fuzzy set theory are reviewed which are obtained from Dubois and Prade (1978) and; Kaufmann and Gupta (1991).

Definition 2.1. *Let X be a universal set; the fuzzy subset $\tilde{A}$ of X with its membership function $\mu_{\tilde{A}}(x): \mathbb{R} \rightarrow [0;1]$ assigns to each element $x \in X$ of real numbers, $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$, where value $\mu_{\tilde{A}}(x)$ represents the grade of membership of x in $\tilde{A}$.*

A fuzzy set A is written as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X, \mu_{\tilde{A}}(x) \in [0,1]\}$.

Definition 2.2.*A fuzzy set $\tilde{A}$ in $X = \mathbb{R}^n$ is a convex fuzzy set if:*

$\forall x_1, x_2 \in X, \forall \lambda \in [0,1]$,

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)). \quad (2.1)$$

Definition 2.3. *Let $\tilde{A}$ be a fuzzy set defined on the set of real numbers $\mathbb{R}$. $\tilde{A}$ is called a normal fuzzy set if there exist $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$.*

Definition 2.4.*A fuzzy number is a normal and convex fuzzy set, with its membership function $\mu_{\tilde{A}}(x)$ defined on the real line $\mathbb{R}$ and piecewise continuous.*

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Definition 2.5. An *LR* fuzzy number is a trapezoidal fuzzy number, if its membership function is given by

$$\mu_{\tilde{m}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \le m, \\ 1, & m \le x \le n \\ R\left(\frac{x-n}{\beta}\right), \text{for } n \le x, \end{cases} \quad (2.2)$$

where $m \le n$, $\alpha, \beta > 0$, and the function $L(.)$ is called a left shape function if the following hold:

i- $L(x) = L(-x)$.
ii- $L(0) = 1, L(1) = 0$.
iii- L is non-increasing on $[0, \infty)$.

Similarly to the right shape function $R(.)$, *LR* fuzzy number is symbolically written $\tilde{m} = (m, n, \alpha, \beta)_{LR}$, where m, n represent the left and right mean values while α *and* β are left and right spreads, respectively. We denote the set of LR fuzzy numbers as $F(\Re)$.

In particular, when $L(x)$ and $R(x)$ are linear functions *L(x) = R(x) = max{0,1 − x}* and $m < n$, fuzzy number $\tilde{m}$ denotes a trapezoidal fuzzy number, $\tilde{m} = (m, n, \alpha, \beta)$. A trapezoidal fuzzy number is called symmetric where $\alpha = \beta$. Also when $m = n$ fuzzy number $\tilde{m}$ denotes a triangular fuzzy number, $\tilde{m} = (m, \alpha, \beta)$

Definition 2.6. *Two fuzzy numbers* $\tilde{m}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\tilde{m}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are called equal, iff* $m_1 = m_2, n_1 = n_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$.

Definition 2.7. *Let* $\tilde{m} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\tilde{n} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *be two LR fuzzy numbers. Then the arithmetic operations on LR fuzzy number are given as follows:*

Addition:
$$\tilde{m} + \tilde{n} = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}. \quad (2.3)$$

Scalar multiplication:
Let $\lambda \in \mathbb{R}$ then,

$$\delta \otimes \tilde{m} = \begin{cases} (\lambda m, \lambda n, \lambda\alpha, \lambda\beta)_{LR} & \lambda \ge 0, \\ (\lambda n, \lambda m, -\lambda\beta, -\lambda\alpha)_{LR} & \lambda < 0. \end{cases} \quad (2.4)$$

Definition 2.8. *Let* $A = (a_{ij})$ *and* $\tilde{B} = (\tilde{b}_{ij})$ *be a* $m \times n$ *and an* $n \times p$ *matrix, respectively. We define* $A \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$, *which is a* $m \times p$ *matrix with*

$$\tilde{c}_{ij} = \sum_{k=1,\ldots,n}^{\oplus} a_{ik} \otimes \tilde{b}_{kj}.$$

Definition 2.9. The $n \times n$ linear system

$$\begin{cases} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \ldots\ldots + a_{1n}\tilde{x}_n = \tilde{b}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \ldots\ldots + a_{2n}\tilde{x}_n = \tilde{b}_2 \\ \vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \ldots\ldots + a_{nn}\tilde{x}_n = \tilde{b}_m. \end{cases}$$

where $A = (a_{ij})$ is an arbitrary crisp matrix, $1 \le i, j \le n$, $\tilde{B} = (\tilde{b}_i) 1 \le i \le n$ is an L-R fuzzy vector, then the system is called LR fuzzy linear system (LR-FLS). The matrix form of the above system is $A\tilde{X} = \tilde{B}$.

SOLUTION OF LR-FLS WITH TRAPEZOIDAL FUZZY NUMBERS

This section discusses the solution of LR fuzzy linear systems with trapezoidal fuzzy numbers, new computational method to solve the LR-FLS. The technique used in Malkawi *et al.* (2015) for solving the unrestricted fully fuzzy linear system is modified in order to produce a linear system which is equivalent to fuzzy system without adding any constraints to the fuzzy system.

To solve the LR-FLS $A \otimes \tilde{X} = \tilde{B}$,
where
$A = (a_{ij})_{n\times n}$, $\tilde{X} = (\tilde{x}_j)_{n\times 1}$, and $\tilde{B}_i = (\tilde{b}_i)_{n\times 1}$,
$\tilde{x}_j = (m_j^x, n_j^x, \alpha_j^x, \beta_j^x)$, and $\tilde{b}_i = (m_i^b, n_i^b, \alpha_i^b, \beta_i^b)$.
The $n \times n$ LR- FLS may be written as

$$\textstyle\sum_{j=1}^{\oplus} a_{ij} \otimes \tilde{x}_j = \tilde{b}_j, \forall i = 1, 2, \ldots, n, \quad (3.1)$$

$$\sum_{j=1}^{\oplus} a_{ij} \otimes (m_j^x, n_{i,j}^x, \alpha_j^x, \beta_j^x) = (m_i^b, n_i^b, \alpha_i^b, \beta_i^b) \quad (3.2)$$

Step 1. Given $a_{i,j}^+, a_{i,j}^-$, let,

$$a_{ij} = \begin{cases} a_{i,j}^+, & a_{ij} \ge 0, \\ a_{i,j}^-, & a_{ij} < 0, \end{cases} \quad (3.3)$$

clearly, either $a_{i,j}^+ = 0$ or $a_{i,j}^- = 0$ (or both).
Then, $a_{ij} = a_{i,j}^+ + a_{i,j}^-$, and $(a_{i,j}^+)(a_{i,j}^-) = 0$.

Step 2. Assuming m_{ij}^a and n_{ij}^a according to the sign of a_{ij}.

To find the mean values m_j^x and n_j^x are written in piecewise function using (3.3).

$$m_{ij}^a = \begin{cases} a_{i,j}^+ m_j^x, & a_{ij} \ge 0, \\ a_{i,j}^- n_j^x, & a_{ij} < 0. \end{cases} \quad (3.4)$$

Thus,
$$m_{ij}^a = a_{i,j}^+ m_j^x + a_{i,j}^- n_j^x, \quad (3.5)$$
because $(a_{i,j}^+ m_j^x)(a_{i,j}^- n_j^x) = 0$.
Similarly,

$$n_{ij}^a = \begin{cases} a_{i,j}^+ n_j^x, & a_{ij} \ge 0, \\ a_{i,j}^- m_j^x, & a_{ij} < 0, \end{cases} \quad (3.6)$$

then
$$n_{ij}^a = a_{i,j}^+ n_j^x + a_{i,j}^- m_j^x, \quad (3.7)$$
because $(a_{i,j}^+ m_j^x)(a_{i,j}^- n_j^x) = 0$.

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Step 3. Assuming $\alpha^a_{i,j}$ and $\beta^a_{i,j}$ according to the sign of a_{ij}.

To find the mean values α^x_j and β^x_j are written in piecewise function using (3.3).

$$\alpha^a_{i,j} = \begin{cases} a^+_{i,j}\alpha^x_j, & a_{ij} \geq 0, \\ -a^-_{i,j}\beta^x_j, & a_{ij} < 0, \end{cases} \quad (3.8)$$

then

$$\alpha^a_{i,j} = a^+_{i,j}\alpha^x_j - a^-_{i,j}\,\beta^a_{i,j} \quad (3.9)$$

because $\left(-a^+_{i,j}\alpha^x_j\right)\left(a^-_{i,j}\,\beta^a_{i,j}\right) = 0$.

Similarly,

$$\beta^a_{i,j} = \begin{cases} a^+_{i,j}\beta^x_j, & a^+_{i,j} \geq 0, \\ -a^-_{i,j}\alpha^x_j, & a^-_{i,j} < 0, \end{cases} \quad (3.10)$$

therefore,

$$\beta^a_{i,j} = a^+_{i,j}\beta^x_j - a^-_{i,j}\alpha^x_j \quad (3.11)$$

because $\left(a^+_{i,j}\beta^x_j\right)\left(-a^-_{i,j}\alpha^x_j\right) = 0$.

Step 4. Write $a_{ij} \otimes \tilde{x}_j$ in (3.1) as piecewise function using (3.4), (3.6), (3.8) and (3.10),

$$a_{ij} \otimes \tilde{x}_j = \left(m^a_{ij}, n^a_{ij}, \alpha^a_{i,j}, \beta^a_{i,j}\right) = \begin{cases} \left(a^+_{i,j}m^x_j, a^+_{i,j}n^x_j,\ a^+_{i,j}\alpha^x_j,\ a^+_{i,j}\beta^x_j\right), & a_{ij} \geq 0, \\ \left(a^-_{i,j}n^x_j, a^-_{i,j}m^x_j, -a^-_{i,j}\beta^x_j, -a^-_{i,j}\alpha^x_j\right), & a_{ij} < 0 \end{cases} \quad (3.12)$$

Using (3.5), (3.7), (3.9), and (3.11), $a_{ij} \otimes \tilde{x}_j$ in (3.12) may be written by two LR fuzzy numbers, where at least one of them is zero,

$$a_{ij} \otimes \tilde{x}_j = \left(a^+_{i,j}m^x_j, a^+_{i,j}n^x_j,\ a^+_{i,j}\alpha^x_j,\ a^+_{i,j}\beta^x_j\right) \oplus \left(a^-_{i,j}n^x_j, a^-_{i,j}m^x_j, -a^-_{i,j}\beta^x_j, -a^-_{i,j}\alpha^x_j\right),$$

then using (2.2),

$$a_{ij} \otimes \tilde{x}_j = \left(a^+_{i,j}m^x_j + a^-_{i,j}n^x_j, a^+_{i,j}n^x_j + a^-_{i,j}m^x_j,\ a^+_{i,j}\alpha^x_j + -a^-_{i,j}\beta^x_j,\ a^+_{i,j}\beta^x_j + -a^-_{i,j}\alpha^x_j\right) \quad (3.13)$$

Step 5. $A \otimes \tilde{X} = \tilde{B}$ may be represented using (3.2) and (3.13):

$$\sum_{j=1}^{n}\left(a^+_{i,j}m^x_j + a^-_{i,j}n^x_j, a^+_{i,j}n^x_j + a^-_{i,j}m^x_j,\ a^+_{i,j}\alpha^x_j \pm a^-_{i,j}\beta^x_j,\ a^+_{i,j}\beta^x_j + -a^-_{i,j}\alpha^x_j\right) = \left(m^b_i, n^b_i, \alpha^b_i, \beta^b_i\right), \quad (3.14)$$

$\forall i = 1, 2, \ldots, n$.

We obtain the following four $n \times n$ linear system:

$$\sum_{j=1}^{n}\left(a^+_{i,j}m^x_j + a^-_{i,j}n^x_j\right) = m^b_i, \quad (3.15a)$$

$$\sum_{j=1}^{n}\left(a^+_{i,j}n^x_j + a^-_{i,j}m^x_j\right) = n^b_i, \quad (3.15b)$$

$$\sum_{j=1}^{n}\left(a^+_{i,j}\alpha^x_j - a^-_{i,j}\beta^x_j\right) = \alpha^b_i, \quad (3.15c)$$

$$\sum_{j=1}^{n}\left(a^+_{i,j}\beta^x_j - a^-_{i,j}\alpha^x_j\right) = \beta^b_i. \quad (3.15d)$$

Collect the above linear systems into $4n \times 4n$ linear system, then solve it to find $m^x_j, n^x_{i,j}, \alpha^x_j$ and $\beta^x_j, \forall j = 1, 2, \ldots, n$.

The next definition and theorem show that the LR-FLS is equivalent linear system.

Definition 3.1. Consider the matrix $A = \left(a_{ij}\right)_{n\times n}$ we define the matrices P and N according to the sign of a_{ij} is as follows,

$P = \left(a^+_{i,j}\right)_{n\times n}, N = \left(a^-_{i,j}\right)_{n\times n}$.

The following theorem finds the solution of LR-FLS using the matrices P and N.

Theorem 3.1. Consider the following two independence crisp linear systems. Then the solutions of LR-FLS are given as follows:

i. The mean values m^x and n^x can be obtained using the following $2n \times 2n$ linear system:

$$\begin{cases} Pm^x + Nn^x = m^b, \\ Pn^x + Nm^x = n^b, \end{cases} \quad (3.16)$$

ii. The spread values α^x and β^x can be obtained using the following $2n \times 2n$ linear system:

$$\begin{cases} P\alpha^x - N\beta^x = \alpha^b, \\ P\beta^x - N\alpha^x = \beta^b, \end{cases} \quad (3.17)$$

where,

$P = \left(a^+_{i,j}\right)_{n\times n}, N = \left(a^-_{i,j}\right)_{n\times n}$.

Proof.

i. To find the mean values, using (3.15a),

$$\sum_{j=1}^{n}\left(a^+_{i,j}m^x_j + a^-_{i,j}n^x_j\right) = \sum_{j=1}^{n}a^+_{i,j}m^x_j + \sum_{j=1}^{n}a^-_{i,j}n^x_j = m^b_i,$$

then,

$P\,m^x + N\,n^x = m^b$.

Similarly, using (3.15b),

$$\sum_{j=1}^{n}\left(a^+_{i,j}n^x_j + a^-_{i,j}m^x_j\right) = \sum_{j=1}^{n}a^+_{i,j}n^x_j + \sum_{j=1}^{n}a^-_{i,j}m^x_j = n^b_i, \forall i = 1, 2, \ldots, n,$$

then,

$P\,n^x + Nm^x = n^b_i$

which provides the following system linear system:

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$$\begin{cases} P\,m^x + N\,n^x = m^b, \\ P\,n^x + N\,m^x = n^b. \end{cases}$$

The mean values m^x and n^x are obtained using the above system.

ii. To find the spreads, using $(3.15c)$,

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}\alpha_j^x - a_{i,j}^{-}\beta_j^x \right) = \sum_{j=1}^{n} a_{i,j}^{+}\alpha_j^x - \sum_{j=1}^{n} a_{i,j}^{-}\beta_j^x = \alpha_i^b,\ \ \forall i = 1, 2, \dots, n,$$

then,

$$P\,\alpha^x - N\,\beta^x = \alpha^b.$$

Similarly, using $(3.15d)$,

$$\sum_{j=1}^{n}\left(a_{i,j}^{+}\beta_j^x - a_{i,j}^{-}\alpha_j^x \right) = \sum_{j=1}^{n} a_{i,j}^{+}\beta_j^x - \sum_{j=1}^{n} -a_{i,j}^{-}\alpha_j^x = \beta_i^b,$$

$\forall i = 1, 2, \dots, n,$

$$P\,\beta^x - N\,\alpha^x = \beta^b,$$

then

$$\begin{cases} P\,\alpha^x - N\,\beta^x = \alpha^b, \\ P\,\beta^x - N\,\alpha^x = \beta^b. \end{cases}$$

The spreads values α^x and β^x are obtained using the above system. The definition of associated linear system for solvingFFLS in Malkawi *et al.* (2014b) is modified to construct linear system is equivalent to LR-FLS.

Definition 3.2. The associated linear system (LR-ALS) of LR-FLS is defined as follows:

$GX = B$

$$\begin{pmatrix} P & N & 0 & 0 \\ N & P & 0 & 0 \\ 0 & 0 & P & -N \\ 0 & 0 & -N & P \end{pmatrix} \begin{pmatrix} m^x \\ n^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^b \\ n^b \\ \alpha^b \\ \beta^b \end{pmatrix}. \qquad (3.18)$$

The following theorem shows the relation between the solution of linear system and LR-FLS.

Theorem 3.2. The unique crisp vectors solution $m^x, n^x, \alpha^x,$ andβ^x of Xin LR-ALS and fuzzy solution $\tilde{X}$ of LR-FLS is equivalent.

Proof.

Using (3.18) and matrix multiplication, then the following equations

$$\begin{cases} P\,m^x + N\,n^x + 0\alpha^x + 0\beta^x = m^b, \\ P\,n^x + N\,m^x + 0\alpha^x + 0\beta^x = n^b, \\ 0\,m^x + 0\,n^x + P\alpha^x - N\beta^x = \alpha^b, \\ 0\,m^x + 0\,n^x - N\alpha^x + P\beta^x = \alpha^b \end{cases} \longrightarrow \begin{cases} P\,m^x + N\,n^x = m^b, \\ P\,n^x + N\,m^x = n^b, \\ P\alpha^x - N\beta^x = \alpha^b, \\ P\beta^x - N\alpha^x = \alpha^b \end{cases}$$

are obtained which are equivalent to $(3.16), (3.17)$.■

The following numerical examples are illustrated to show the ability of the proposed method in obtaining the unique solution for LR-FLS. We verify our method using Example 3.1. in Nasseri and Gholami (2011).

Example 3.1. Considering the following 3×3 LR-FLS (Nasseri and Gholami, 2011),

$$\begin{cases} 1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus 1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ \quad \oplus -1 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) = (6, 11, 4, 4), \\ 1 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus -2 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ \quad \oplus 1 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) = (-11, -5, 4, 7), \\ 2 \otimes (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \oplus 1 \otimes (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ \quad \oplus 3 \otimes (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) = (-7, 4, 7, 8). \end{cases}$$

where$\tilde{x}_i = (m_i{}^x, n_i{}^x, \alpha_i{}^x, \beta_i{}^x), i = 1, \dots, 3$ are arbitrary trapezoidal fuzzy numbers.

Solution

The LR-FLS may be written in the following matrix form:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} (6, 11, 4, 4) \\ (-11, -5, 4, 7) \\ (-7, 4, 7, 8) \end{pmatrix}.$$

The crisp matrices P and$N$ are defined using crisp matrix A, as follows:

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The block crisp matrix G is then constructed as follows:

$$G = \begin{pmatrix} P & N & 0 & 0 \\ N & P & 0 & 0 \\ 0 & 0 & P & -N \\ 0 & 0 & -N & P \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 3 \end{pmatrix},$$

$$|G| = 169,$$

then the system has a unique solution since $|G| \neq 0$.

The crisp vectors $m^x, n^x, \alpha^x,$ and β^x are made using fuzzy vector $\tilde{X}$to construct the crisp block vector X.

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Then $X=\begin{pmatrix} m^x \\ n^x \\ \alpha^x \\ \beta^x \end{pmatrix}=\begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} n_1^x \\ n_2^x \\ n_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix}.$

The crisp vectors m^b, n^b, α^b and β^b are made using fuzzy vector $\tilde{B}$.

$$m^b=\begin{pmatrix} m_1^b \\ m_2^b \\ m_3^b \end{pmatrix}=\begin{pmatrix} 6 \\ -11 \\ -7 \end{pmatrix}, n^b=\begin{pmatrix} n_1^b \\ n_2^b \\ n_3^b \end{pmatrix}=\begin{pmatrix} 11 \\ -5 \\ 4 \end{pmatrix},$$

$$, \alpha^b=\begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \alpha_3^b \end{pmatrix}=\begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix}, \beta^b=\begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \beta_3^b \end{pmatrix}=\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix},$$

The crisp vector B is constructed as follows:

$$B=\begin{pmatrix} m^b \\ n^b \\ \alpha^b \\ \beta^b \end{pmatrix}=\begin{pmatrix} \begin{pmatrix} m_1^b \\ m_2^b \\ m_3^b \end{pmatrix} \\ \begin{pmatrix} n_1^b \\ n_2^b \\ n_3^b \end{pmatrix} \\ \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \\ \alpha_3^b \end{pmatrix} \\ \begin{pmatrix} \beta_1^b \\ \beta_2^b \\ \beta_3^b \end{pmatrix} \end{pmatrix}=\begin{pmatrix} \begin{pmatrix} 6 \\ -11 \\ -7 \end{pmatrix} \\ \begin{pmatrix} 11 \\ -5 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 4 \\ 4 \\ 7 \end{pmatrix} \\ \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix} \end{pmatrix}=\begin{pmatrix} 6 \\ -11 \\ -7 \\ 11 \\ -5 \\ 4 \\ 4 \\ 4 \\ 7 \\ 4 \\ 7 \\ 8 \end{pmatrix}.$$

Thus, the original LR-FLS is equivalent to the following linear system $GX=B$:

Then,

$$X=G^{-1}B=\begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \\ n_1^x \\ n_2^x \\ n_3^x \\ \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \\ \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix}=\begin{pmatrix} 1 \\ 3 \\ -4 \\ 3 \\ 4 \\ -2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, X=\begin{pmatrix} \begin{pmatrix} m_1^x \\ m_2^x \\ m_3^x \end{pmatrix} \\ \begin{pmatrix} n_1^x \\ n_2^x \\ n_3^x \end{pmatrix} \\ \begin{pmatrix} \alpha_1^x \\ \alpha_2^x \\ \alpha_3^x \end{pmatrix} \\ \begin{pmatrix} \beta_1^x \\ \beta_2^x \\ \beta_3^x \end{pmatrix} \end{pmatrix}=\begin{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \end{pmatrix}.$$

By partitioning X according to vectors m^x, n^x, α^x, and β^x, we find the fuzzy solution $\tilde{X}$,

$$\tilde{X}=\begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix}=\begin{pmatrix} (1,3,1,2) \\ (3,4,2,1) \\ (-4,-2,1,1) \end{pmatrix}.$$

Example 3.2. Consider the following 3×3 LR-FLS (Allahviranloo *et al.*, (2012a)):

$$\begin{pmatrix} -1 & -1 & 1 \\ 2 & 1 & -2 \\ -3 & 2 & -1 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix}=\begin{pmatrix} (2,7,6,6) \\ (-7,1,11,11) \\ (-12,-3,11,11) \end{pmatrix}.$$

where $\tilde{x}_i=(m_i{}^x, n_i{}^x, \alpha_i{}^x, \beta_i{}^x)$, $i=1,\ldots,3$ are arbitrary trapezoidal fuzzy numbers.

Solution

According to Allahviranloo *et al.* (2012b), the fuzzy exact solution for LR-FLS is

$$\tilde{X}_v=\begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix}=\begin{pmatrix} (1,2,2,2) \\ (-1,1,1,1) \\ (2,4,3,3) \end{pmatrix}.$$

Checking $m_1^b=2$ in the right hand side,

$$\tilde{B}=\begin{pmatrix} (m_1^b, n_1^b, \alpha_1^b, \beta_1^b) \\ (m_2^b, n_2^b, \alpha_2^b, \beta_2^b) \\ (m_3^b, n_3^b, \alpha_3^b, \beta_3^b) \end{pmatrix}=\begin{pmatrix} (2,7,6,6) \\ (-7,1,11,11) \\ (-12,-3,11,11) \end{pmatrix}$$

$m_1^b=(-1)(n_1^x)+(-1)(n_2^x)+(1)(m_3^x)=$
$(-1)(2)+(-1)(1)+(1)(2)=-1,$

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The system has a unique solution; $|G| = 64$, hence $|G| \neq 0$. the exact solution is obtained using $GX = B$in (3.18),

$$\tilde{X}_g = \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \end{pmatrix} = \begin{pmatrix} \left(-\frac{5}{4}, -\frac{1}{4}, 2,2\right) \\ (-7, -5,1,1) \\ \left(-\frac{13}{4}, -\frac{5}{4}, 3,3\right) \end{pmatrix}.$$

As compared to the Allahviranloo *et al*. (2012b), his solution did not satisfy the fuzzy system which has been discussed in Malkawi (2014.

CONCLUSIONS

The LR-FLS is solved by transforming the system to an equivalent linear system where the solution is obtained without using any fuzzy operation. From the results, it show that this new developed method required less computation time and able to solve large scale of system as can be seen in the example 3.3 (appendix).

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APPENDIX

$Examples\ 3.3.$Consider the following 10×10 LR FLS.

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$$\begin{pmatrix} -3 & 6 & 7 & 8 & 4 & 2 & -3 & 4 & 1 & 5 \\ 2 & 3 & 5 & 7 & 2 & 7 & 9 & 0 & -4 & -1 \\ -1 & -2 & 8 & 5 & -6 & -3 & 8 & 1 & 5 & 6 \\ 7 & 8 & 4 & 8 & -3 & 6 & 7 & 8 & 4 & 4 \\ 6 & 7 & -8 & 4 & 7 & 9 & 0 & 3 & 2 & 7 \\ 7 & -3 & 6 & 7 & -7 & 1 & 4 & 7 & 9 & 0 \\ 7 & 9 & 0 & 5 & 7 & 5 & 0 & 3 & 3 & -7 \\ 5 & 7 & 8 & -4 & 2 & 3 & 7 & 4 & 4 & 2 \\ 1 & 7 & 9 & 0 & -3 & 0 & 7 & 2 & 2 & 7 \\ -3 & -6 & 5 & 7 & 0 & 4 & -3 & 7 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} (m_1^x, n_1^x, \alpha_1^x, \beta_1^x) \\ (m_2^x, n_2^x, \alpha_2^x, \beta_2^x) \\ (m_3^x, n_3^x, \alpha_3^x, \beta_3^x) \\ (m_4^x, n_4^x, \alpha_4^x, \beta_4^x) \\ (m_5^x, n_5^x, \alpha_5^x, \beta_5^x) \\ (m_6^x, n_6^x, \alpha_6^x, \beta_6^x) \\ (m_7^x, n_7^x, \alpha_7^x, \beta_7^x) \\ (m_8^x, n_8^x, \alpha_8^x, \beta_8^x) \\ (m_9^x, n_9^x, \alpha_9^x, \beta_9^x) \\ (m_{10}^x, n_{10}^x, \alpha_{10}^x, \beta_{10}^x) \end{pmatrix} = \begin{pmatrix} (-167, 23, 122, 226) \\ (-136, 80, 174, 120) \\ (-177, 22, 128, 199) \\ (-168, 98, 224, 253) \\ (51, 207, 216, 151) \\ (-222, 13, 197, 273) \\ (-83, 90, 205, 148) \\ (-145, 51, 216, 184) \\ (-109, 13, 140, 167) \\ (-205, -49, 96, 231) \end{pmatrix}.$$

where$\tilde{x}_i = ({m_i}^x, {n_i}^x, {\alpha_i}^x, {\beta_i}^x),\ i = 1, \ldots, 10.$ are arbitrary trapezoidal fuzzy numbers.

Solution

The LR FLS may be written in matrix form is as follows:

The crisp matrices P, N are defined using A is as follws:

$$P = \begin{pmatrix} 0 & 6 & 7 & 8 & 4 & 2 & 0 & 4 & 1 & 5 \\ 2 & 3 & 5 & 7 & 2 & 7 & 9 & 0 & 0 & 0 \\ 0 & 0 & 8 & 5 & 0 & 0 & 8 & 1 & 5 & 6 \\ 7 & 8 & 4 & 8 & 0 & 6 & 7 & 8 & 4 & 4 \\ 6 & 7 & 0 & 4 & 7 & 9 & 0 & 3 & 2 & 7 \\ 7 & 0 & 6 & 7 & 0 & 1 & 4 & 7 & 9 & 0 \\ 7 & 9 & 0 & 5 & 7 & 5 & 0 & 3 & 3 & 0 \\ 5 & 7 & 8 & 0 & 2 & 3 & 7 & 4 & 4 & 2 \\ 1 & 7 & 9 & 0 & 0 & 0 & 7 & 2 & 2 & 7 \\ 0 & 0 & 5 & 7 & 0 & 4 & 0 & 7 & 2 & 0 \end{pmatrix},$$

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$$N=\begin{pmatrix}-3&0&0&0&0&0&-3&0&0&0\\0&0&0&0&0&0&0&0&-4&-1\\-1&-2&0&0&-6&-3&0&0&0&0\\0&0&0&0&-3&0&0&0&0&0\\0&0&-8&0&0&0&0&0&0&0\\0&-3&0&0&-7&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&-7\\0&0&0&-4&0&0&0&0&0&0\\0&0&0&0&-3&0&0&0&0&0\\-3&-6&0&0&0&0&-3&0&0&0\end{pmatrix}.$$

The crisp vectors m^b, n^b, α^b andβ^b are made using fuzzy vector $\tilde{B}$,

$$m^b=\begin{pmatrix}m_1^b\\m_2^b\\m_3^b\\m_4^b\\m_5^b\\m_6^b\\m_7^b\\m_8^b\\m_9^b\\m_{10}^b\end{pmatrix}=\begin{pmatrix}-167\\-136\\-177\\-168\\51\\-222\\-83\\-145\\-109\\-205\end{pmatrix}, n^b=\begin{pmatrix}n_1^b\\n_2^b\\n_3^b\\n_4^b\\n_5^b\\n_6^b\\n_7^b\\n_8^b\\n_9^b\\n_{10}^b\end{pmatrix}=\begin{pmatrix}23\\80\\22\\98\\207\\13\\90\\51\\13\\-49\end{pmatrix}, \alpha^b=\begin{pmatrix}\alpha_1^b\\\alpha_2^b\\\alpha_3^b\\\alpha_4^b\\\alpha_5^b\\\alpha_6^b\\\alpha_7^b\\\alpha_8^b\\\alpha_9^b\\\alpha_{10}^b\end{pmatrix}=\begin{pmatrix}122\\174\\128\\224\\216\\197\\205\\216\\140\\96\end{pmatrix}, \beta^b=\begin{pmatrix}\beta_1^b\\\beta_2^b\\\beta_3^b\\\beta_4^b\\\beta_5^b\\\beta_6^b\\\beta_7^b\\\beta_8^b\\\beta_9^b\\\beta_{10}^b\end{pmatrix}=\begin{pmatrix}226\\120\\199\\253\\151\\273\\148\\184\\167\\231\end{pmatrix}.$$

Hence, the equivalent linear system for LR FLS is constructed using (3.20), $GX=B$,

$$\begin{pmatrix}P&N&0&0\\N&P&0&0\\0&0&P&-N\\0&0&-N&P\end{pmatrix}\begin{pmatrix}m^x\\n^x\\\alpha^x\\\beta^x\end{pmatrix}=\begin{pmatrix}m^b\\n^b\\\alpha^b\\\beta^b\end{pmatrix},$$

Hence, The fuzzy solution is

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$$\tilde{X}=\begin{pmatrix}(m_1^x,n_1^x,\alpha_1^x,\beta_1^x)\\(m_2^x,n_2^x,\alpha_2^x,\beta_2^x)\\(m_3^x,n_3^x,\alpha_3^x,\beta_3^x)\\(m_4^x,n_4^x,\alpha_4^x,\beta_4^x)\\(m_5^x,n_5^x,\alpha_5^x,\beta_5^x)\\(m_6^x,n_6^x,\alpha_6^x,\beta_6^x)\\(m_7^x,n_7^x,\alpha_7^x,\beta_7^x)\\(m_8^x,n_8^x,\alpha_8^x,\beta_8^x)\\(m_9^x,n_9^x,\alpha_9^x,\beta_9^x)\\(m_{10}^x,n_{10}^x,\alpha_{10}^x,\beta_{10}^x)\end{pmatrix}=\begin{pmatrix}(-1,8,9,0)\\(2,3,7,7)\\(-10,-9,3,6)\\(-7,5,3,6)\\(0,3,6,1)\\(0,0,0,0)\\(-4,6,6,1)\\(-6,-6,1,8)\\(-2,0,4,8)\\(3,5,0,1)\end{pmatrix}.$$