A Note On "SOLVING FULLY FUZZY LINEAR SYSTEMS BY USING IMPLICIT GAUSS–CHOLESKY ALGORITHM"

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This paper shows that the exact solutions given for the fully fuzzy linear systems presented as examples in [1] are non-fuzzy solutions. That is, the proposed solutions in [1] do not correspond to systems. For this reason, we provide approximate fuzzy solutions for all examples. Finally, we demonstrate the efficiency of these approximate solutions using the distance metric function presented in [13].

Keywords: fully fuzzy linear system; fuzzy number; triangular fuzzy number; fuzzy number vector solution

1. Preliminaries

We first review some basic definitions of fuzzy theory that will be used in our following arguments:

Definition 1.1. *Let X be a universal set. We define the fuzzy subset $\tilde{A}$ of X by its membership function $\mu_{\tilde{A}}(x)\colon \mathbb{R} \to [0;1]$, which assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$. The value $\mu_{\tilde{A}}(x)$ represents the grade of membership of x in $\tilde{A}$.*
A fuzzy set A is written as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X, \mu_{\tilde{A}}(x) \in [0,1]\}$.

Definition 1.2. *A fuzzy set $\tilde{A}$ in $X = \mathbb{R}^n$ is said to be convex if:*

$$\forall x_1, x_2 \in X, \forall \lambda \in [0,1],$$
$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min\big(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\big).$$

Definition 1.3. *Let $\tilde{A}$ be a fuzzy set defined on the set of real numbers $\mathbb{R}$. $\tilde{A}$ is called a normal fuzzy set if there exists some $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$.*

Definition 1.4. *A fuzzy number is a normal and convex fuzzy set, and its membership function $\mu_{\tilde{A}}(x)$ is defined on the real line $\mathbb{R}$ and is piecewise continuous.*

Definition 1.5. *(LR Fuzzy number) A fuzzy number $\widetilde{m}$ is called an LR fuzzy number if its membership function is defined as follows:*

$$\mu_{\widetilde{m}}(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & for \quad x \leq m,\ \alpha > 0, \\ R\left(\dfrac{x-m}{\beta}\right) & for \quad m \leq x,\ \beta > 0, \end{cases}$$

where $m, \alpha, \beta \in \mathbb{R}$.
The function $L(.)$ is called a left shape function if the following hold:

1- $L(x) = L(-x)$,
2- $L(0) = 1, L(1) = 0$,
3- *L is non-increasing on $[0, \infty]$.*

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The definition of the right shape function $R(.)$is similar to that of $L(.)$. An LR fuzzy number is written symbolically as $\widetilde{m} = (m, \alpha, \beta)_{LR}$, where m represents the mean value and α, β are the left and right spreads, respectively. That is, α and β are the coefficients of "fuzziness," and as the spreads increase, $\widetilde{m}$ becomes increasingly fuzzier.

Definition 1.6. *An LR fuzzy number $\widetilde{m} = (m, \alpha, \beta)_{LR}$ is called a triangular fuzzy number if $L = R = max(0,1 - x)$.*
The set of triangular fuzzy numbers is denoted by $F(\Re)$.

Definition 1.7. *(Arithmetic operations on LR fuzzy numbers) We represent arithmetic operations for two LR fuzzy numbers $\widetilde{m} = (m, \alpha, \beta)_{LR}$ and $\tilde{n} = (n, \gamma, \delta)_{LR}$ as follows:*
Addition:

$$(m, \alpha, \beta)_{LR} \oplus \ (n, \gamma, \delta)_{LR} = \ (m + n, \alpha + \gamma, \beta + \delta)_{LR}$$

Opposite:

$$-\ (m, \alpha, \beta)_{LR} \ = (-m, \beta, \alpha)_{RL}$$

Subtraction:

$$(m, \alpha, \beta)_{LR} \ominus \ (n, \gamma, \delta)_{RL} = \ (m - n, \alpha + \delta, \beta + \gamma)_{LR}$$

Approximated multiplication operation of two fuzzy numbers:

i- If $\widetilde{m} > 0$ and $\tilde{n} > 0$, then

$$(m, \alpha, \beta)_{LR} \otimes \ (n, \gamma, \delta)_{LR} \cong \ (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}.$$

ii- If $\widetilde{m} < 0$ and $\tilde{n} > 0$, then

$$(m, \alpha, \beta)_{RL} \otimes \ (n, \gamma, \delta)_{LR} \cong \ (mn, n\alpha - m\delta, n\beta - m\gamma)_{RL}.$$

iii- If $\widetilde{m} < 0$ and $\tilde{n} < 0$, then

$$(m, \alpha, \beta)_{LR} \otimes \ (n, \gamma, \delta)_{LR} \cong \ (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{RL}.$$

Scalar multiplication:
For $\lambda \in R$,

$$\delta \otimes (m, \alpha, \beta)_{LR} = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta)_{LR}, & \lambda \geq 0, \\ (\lambda m, -\lambda\beta, -\lambda\alpha)_{RL,} & \lambda < 0. \end{cases}$$

Definition 1.8. *A matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ is called a fuzzy matrix if $\tilde{a}_{ij} \in \ F(\Re), \forall i, j = 1, \dots, n$. A fuzzy matrix $\tilde{A}$ will be non-negative (non-positive) and denoted by $\tilde{A} \geq 0$ ($\tilde{A} \leq 0$) if each element $\tilde{a}_{ij}$ is non-negative (non-positive).*

Definition 1.9. *A vector $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ is called a fuzzy vector if $\tilde{x}_i \in \ F(\Re), \forall i = 1, \dots, n$.*

Definition 1.10. *Let $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ be $m \times n$ and $n \times p$ matrices, respectively. We define $\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$ as the $m \times p$ matrix where*

$$\tilde{c}_{ij} = \sum_{k=1,\dots,n}^{\oplus} \tilde{a}_{ik} \otimes \tilde{b}_{kj}.$$

Definition 1.11. *(Fully fuzzy linear system) Consider the $n \times n$ linear system of equations:*

$$\begin{cases} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \ldots\ldots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \ldots\ldots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ \vdots \\ \vdots \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \ldots\ldots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n. \end{cases}$$

The matrix form of the above equations is

$$\tilde{A} \otimes \tilde{x} = \tilde{b}$$

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{pmatrix}.$$

This system is called a fully fuzzy linear system (FFLS) if the coefficient matrix $\tilde{A} = (\tilde{a}_{ij}),\ 1 \leq i,j \leq n$, is a fuzzy matrix, $\tilde{x}$ and $\tilde{b}$ are fuzzy vectors, and $\tilde{x}$ is an unknown to be found as the solution.

Remark 1.1. *To find the nearest solution, the following distance metric for triangular fuzzy numbers can be used [13].*

For the triangular fuzzy numbers $\tilde{a} = (a, \alpha, \beta)$ and $\tilde{b} = (b, \gamma, \eta)$, Ming et al. [15] introduced the distance function:

$$D_2^2(\tilde{a}, \tilde{b}) = \left(\frac{1}{2}\right)(4(a-b)^2 + (\alpha-\gamma)^2 + (\beta-\eta)^2) + (a-b)(\gamma+\eta-\alpha-\beta).$$

For two LR fuzzy vectors $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$, $\tilde{Y} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$, this is defined as:

$$D_n^2(\tilde{X}, \tilde{Y}) = \sum_{i=1}^{n} D_2^2(\tilde{x}_i, \tilde{y}_i).$$

2. Numerical Examples

In this section, we discuss the solutions proposed in [1], and show that they do not correspond to systems by applying arithmetic operations on LR fuzzy numbers to prove $\tilde{A}\tilde{X}_a, \neq \tilde{B}$, where $\tilde{X}_a$ is the solution proposed in [1]. Using the distance metric function in [15], we also show that $D_n(\tilde{A}\tilde{X}_a, \tilde{B}) \neq 0$.

Exact solutions are provided using the methods in [10,11] to demonstrate that the proposed FFLSs do not have exact fuzzy solutions. Therefore, approximate fuzzy solutions are provided, and the distance metric function in [15] allows us to show that the resulting solutions are favorable because $D_n(\tilde{A}\tilde{X}_g, \tilde{B})$ is very small compared with $D_n(\tilde{A}\tilde{X}_a, \tilde{B})$, where $\tilde{X}_g$ is the provided solution.

Example 2.1. Consider the following FFLS [1]:

$$\begin{cases} (4,3,2) \otimes (x_1, y_1, z_1) \oplus (5,2,1) \otimes (x_2, y_2, z_2) \oplus (3,0,3) \otimes (x_3, y_3, z_3) = (71, 54, 76), \\ (7,4,3) \otimes (x_1, y_1, z_1) \oplus (10,6,3) \otimes (x_2, y_2, z_2) \oplus (2,1,1) \otimes (x_3, y_3, z_3) = (118, 115, 129), \\ (6,2,2) \otimes (x_1, y_1, z_1) \oplus (7,1,2) \otimes (x_2, y_2, z_2) \oplus (15,5,4) \otimes (x_3, y_3, z_3) = (155, 89, 151). \end{cases}$$

According to [1], the exact fuzzy solution for this FFLS is

$$\tilde{X}_a = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (4,2,2) \\ (8,3,5) \\ (5,1,4) \end{pmatrix}.$$

We may note that $\tilde{A}\tilde{X}_a \neq \tilde{B}$, as

$$\tilde{A}\tilde{X}_a = \begin{pmatrix} (71,54,76) \\ (118,115,113) \\ (155,89,151) \end{pmatrix}.$$

Moreover, from Remark 1.1, we have that:

$$D_3\left(\tilde{A}\tilde{X}_a, \tilde{B}\right) = 11.3137.$$

In fact, the exact solution of this system is the following non-fuzzy vector:

$$\tilde{X}_e = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (4, 2, -16.782) \\ (8, 3, 19.608) \\ (5, 1, 4.695) \end{pmatrix},$$

which satisfies

$$D_3\left(\tilde{A}\tilde{X}_e, \tilde{B}\right) = 0.$$

In this case, we have provided an approximate fuzzy solution $\tilde{X}_g$:

$$\tilde{X}_g = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (3.569, 3.569, 0.) \\ (8.294, 1.912, 8.171) \\ (5.034, 0.905, 3.328) \end{pmatrix},$$

such that

$$\tilde{A}\tilde{X}_g = \begin{pmatrix} (70.8544, 53.8544, 81.3822) \\ (118, 115, 129) \\ (155, 89, 151) \end{pmatrix},$$

and

$$D_3\left(\tilde{A}\tilde{X}_g, \tilde{B}\right) = 3.9114.$$

Example 2.2. Consider the following FFLS [1]:

$$\begin{cases} (4,3,2) \otimes (x_1, y_1, z_1) \oplus (5,2,1) \otimes (x_2, y_2, z_2) \oplus (3,0,3) \otimes (x_3, y_3, z_3) = (127, 108, 140), \\ (7,4,3) \otimes (x_1, y_1, z_1) \oplus (10,6,3) \otimes (x_2, y_2, z_2) \oplus (2,1,1) \otimes (x_3, y_3, z_3) = (201, 184, 206), \\ (6,2,2) \otimes (x_1, y_1, z_1) \oplus (7,1,2) \otimes (x_2, y_2, z_2) \oplus (15,5,4) \otimes (x_3, y_3, z_3) = (291, 224, 287). \end{cases}$$

According to [1], the exact fuzzy solution for this FFLS is

$$\tilde{X}_a = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (13, 7, 2) \\ (9, 1, 8) \\ (10, 6, 9) \end{pmatrix}.$$

Again, note that $\tilde{A}\tilde{X}_a \neq \tilde{B}$:

$$\tilde{A}\tilde{X}_a = \begin{pmatrix} (127, 108, 140) \\ (201, 187, 188) \\ (291, 224, 287) \end{pmatrix}.$$

Moreover, Remark 1.1 gives

$$D_3(\tilde{A}\tilde{X}_a, \tilde{B}) = 12.903.$$

In fact, the exact solution to this system is the non-fuzzy vector:

$$\tilde{X}_e = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (13, 10.5217, -19.1304) \\ (9, -1.73913, 24.4348) \\ (10, 5.86957, 9.78261) \end{pmatrix},$$

which satisfies

$$D_3(\tilde{A}\tilde{X}_e, \tilde{B}) = 0.$$

Similar to the previous example, we can provide an approximate fuzzy solution $\tilde{X}_g$:

$$\tilde{X}_g = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (13, 7.978, 0) \\ (9, 0, 11.352) \\ (10, 6.075, 8.235) \end{pmatrix},$$

such that

$$\tilde{A}\tilde{X}_g = \begin{pmatrix} (127., 107.14, 146.471) \\ (201., 184., 206) \\ (291., 224., 287) \end{pmatrix},$$

and

$$D_3(\tilde{A}\tilde{X}_g, \tilde{B}) = 4.6156.$$

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