

The Triad Design of Order 19

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ABSTRACT: In this paper we present a new method to construct the triad design of order 19, $TD(19)$. First we construct the compatible factorization of order 19, $CF(19)$. Then we build the starter of triad design of order 19, $STD(19)$ using interval techniques of the number of triples in the design. Finally by using addition modular 19 to $STD(19)$ we develop $TD(19)$.

Keywords: Compatible factorization, Triad design, Starter.

1 INTRODUCTION

A *triad design* on v objects, denoted by $TD(v)$, is a way of arranging the distinct triples of $\binom{v}{3}$ into v rows such that:

- (i) Row m contains $\frac{v-1}{2}$ triples, among which object m meets every other object precisely once, and contains also other distinct triples;
- (ii) Each triple occurs exactly once in the design;
- (iii) No two elements (entries) occurs together in two or more triples in any row.

$TD(v)$ is a set of distinct triples (3-element) of a v -set of points that deals with counting and listing triples from a graph of v points. It is associated with counting triangles in a graph which is called triangle problems that gained recently much practical importance since they are central in so-called complex network analysis [6, 7, 8].

The construction of $TD(v)$ is based on the compatible factorization of a graph of order v , denoted by $CF(v)$, which is a $v \times \frac{v-1}{2}$ array that satisfies the following conditions: [5, 9]

- (i) The entries in row m form a near-one-factor with focus m .
- (ii) The triples associated with the rows contain no repetitions.

Example 1.1 If $v = 7$, that is a set of 7 points labeled 1, 2, 3, 4, 5, 6, 7, then the compatible factorization $CF(7)$ is illustrated in Table 1. Append C_1 with C_2 , C_1 with C_3 and C_1 with C_4 , we obtain 21 distinct triples in $CF(7)$. Moreover, $TD(7) = CF(7) \cup \overline{CF(7)}$, where $\overline{CF(7)}$ is the completion of $CF(7)$. Clearly, $TD(7)$ consists of $\binom{7}{3} = 35$ distinct triples. These triples are associated with the triangles that can be formed from K_7 , the complete graph of order 7, see figure 1.

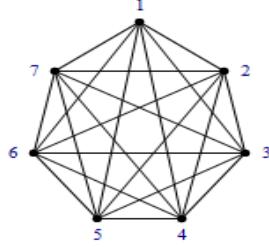
CF(7)				$\overline{CF(7)}$	
C_1	C_2	C_3	C_4		
1	2 7	3 6	4 5	2 3 5	7 6 4
2	3 1	4 7	5 6	3 4 6	1 7 5
3	4 2	5 1	6 7	4 5 7	2 1 6
4	5 3	6 2	7 1	5 6 1	3 2 7
5	6 4	7 3	1 2	6 7 2	4 3 1
6	7 5	1 4	2 3	7 1 3	5 4 2

7	1 6	2 3	3 4	1 2 4	6 5 3
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Table 1. $TD(7)$

Table 1 shows that addition modular 7 to the first row, called starter, produces all the distinct triples.

Once we construct the starter, listing all distinct triples can be done by addition modular v . Our aim in this paper is to present a new method to construct the starter of triad design of order 19, $STD(19)$. This method depends on analysing the triples using interval techniques of the triples in the starter.

Figure 1. Plot of the complete graph K_7

It is known that $TD(v)$ exists if $v \equiv 1 \text{ or } 5 \pmod{6}$ [2, 3]. The construction of $STD(7)$ above is done by the brute force method (trial and error). Moreover, in [1], $STD(13)$ was constructed by using the brute force method and a new method that depends on the interval techniques of the triples in the starter.

In section 2, we study some properties of the starter of triad design on v objects, $STD(v)$. In section 3, we again use the brute force method to construct $STD(19)$ and then develop $TD(19)$. In section 4, we extend the use of the interval techniques of the triples in the starter to build $STD(19)$.

2 PROPERTIES OF THE STARTER

Definition 2.1. The starter of triad design on v objects, $STD(v)$, is the set of triples on v that generates all the triples in $TD(v)$ by addition modular v .

The following lemma provides the number of triples in $STD(v)$ as well as in $\overline{SCF(v)}$, denoted by $|STD(v)|$ and $|\overline{SCF(v)}|$ respectively.

Lemma 2.1 If $v = 6n + 1$, then

$$|STD(v)| = n(6n-1) \text{ and } |\overline{SCF(v)}| = 2n(3n-2).$$

Proof: We prove the first and the proof of the second is the same. The number of triples of $TD(v)$ is equal to $|TD(v)| = \binom{v}{3} = \binom{6n+1}{3} = n(6n+1)(6n-1)$. By the definition of $CF(v)$, the number of rows of $TD(v)$ is equal to $v = 6n + 1$. Hence $|STD(v)| = n(6n-1)$. \square

Each triple in $STD(v)$ consists of three elements (numbers, objects). The first, second and the third element.

Definition 2.2 The r -th elements of $STD(v)$ are the r -th numbers in each triple, denoted by $S_r TD(v)$, for

$SCF(7)$			$\overline{SCF(7)}$	
1 2 7	1 3 6	1 4 5	2 3 5	7 6 4
$\pm 1,$ $\pm 2, \pm 1$	$\pm 2,$ $\pm 3, \pm 2$	$\pm 3,$ $\pm 1, \pm 3$	$\pm 1,$ $\pm 2, \pm 3$	$\pm 1,$ $\pm 2, \pm 3$

$$1 \leq r \leq 3.$$

Example 2.1 From Example 2.1,

$$STD(7) = \{1 2 7; 1 3 6; 1 4 5; 2 3 5; 7 6 4\}. \\ \text{Therefore}$$

$$S_1 TD(7): 1, 1, 1, 2, 7. \quad S_2 TD(7): 2, 3, 4, 3, 6. \\ S_3 TD(7): 7, 6, 5, 5, 4.$$

Table 2. Difference sets for
 $STD(7) = SCF(7) \cup \overline{SCF(7)}$

Note that the difference sets for each triple in $STD(7)$ are listed in table 2 using the facts that $\pm 6 = \pm 1, \pm 5 = \pm 2$ and $\pm 4 = \pm 3$.

From Table 2, each difference occurs 3 times in $SCF(7)$ and 2 times in $\overline{SCF(7)}$, [4].

In the following section, the triples in $STD(19)$ are analyzed in order to construct formulas to generate them and to develop $TD(19)$.

3 ALGORITHM FOR $TD(19)$

In this section, addition modular 19 is used to produce the algorithm for $STD(19)$. Obviously, $STD(19) = SCF(19) \cup \overline{SCF(19)}$. By Lemma 2.1, with $n = 3$, $|STD(19)| = 51$ and $|\overline{SCF(19)}| = 42$. The algorithm for $TD(19)$ is as follows:

Step 1. Generate $SCF(19)$ as shown below.

$$\begin{array}{cccccc} 1 & 2 & 19 & 3 & 18 & 4 & 17 & 5 & 16 \\ 6 & 15 & 7 & 14 & 8 & 13 & 9 & 12 \end{array}$$

10 11. Therefore,

$$SCF(19) = \{1 2 19, 1 3 18, 1 4 17, 1 5 16, 1 6 15, 1 7 14, 1 8 13, 1 9 12, 1 10 11\}.$$

Note that the difference sets for the triples in $SCF(19)$ are listed in table 4 using the facts that $\pm 18 = \pm 1$, $\pm 17 = \pm 2$, $\pm 16 = \pm 3$, $\pm 15 = \pm 4$, $\pm 14 = \pm 5$, $\pm 13 = \pm 6$, $\pm 12 = \pm 7$, $\pm 11 = \pm 8$ and $\pm 10 = \pm 9$.

1	2	19	± 1	± 2	± 1
3	18		± 2	± 4	± 2
4	17		± 3	± 6	± 3
5	16		± 4	± 8	± 4
6	15		± 5	± 9	± 5
7	14		± 6	± 7	± 6
8	13		± 7	± 5	± 7
9	12		± 8	± 3	± 8
10	11		± 9	± 1	± 9

Table 3. Constructing $SCF(19)$

Step 2. Generate $\overline{SCF(19)}$ by using difference set method and addition modular 13.

Note that from Table 3 and 4, each difference occurs 3 times in $SCF(19)$ and 14 times in $\overline{SCF(19)}$.

Table 4. Constructing $SCF(19)$

$SCF(19)$ Differences						$\overline{SCF(19)}$ Differences					
2	3	17	± 1	± 5	± 4	19	18	4	± 5	± 1	± 4
2	4	16	± 2	± 7	± 5	19	17	5	± 7	± 2	± 5
2	5	15	± 3	± 9	± 6	19	16	6	± 9	± 3	± 6
2	6	14	± 4	± 8	± 7	19	15	7	± 8	± 4	± 7
2	7	13	± 5	± 6	± 8	19	14	8	± 6	± 5	± 8
2	8	12	± 6	± 4	± 9	19	13	9	± 4	± 6	± 9
2	9	11	± 7	± 2	± 9	19	12	10	± 2	± 7	± 9
3	4	15	± 1	± 8	± 7	18	17	6	± 8	± 1	± 7
3	5	14	± 2	± 9	± 8	18	16	7	± 9	± 2	± 8
3	6	13	± 3	± 7	± 9	18	15	8	± 7	± 3	± 9
3	7	12	± 4	± 5	± 9	18	14	9	± 5	± 4	± 9
3	8	11	± 5	± 3	± 8	18	13	10	± 3	± 5	± 8
4	5	13	± 1	± 8	± 9	17	16	8	± 8	± 1	± 9
4	6	12	± 2	± 6	± 8	17	15	9	± 6	± 2	± 8
4	7	11	± 3	± 4	± 7	17	14	10	± 4	± 3	± 7
5	6	12	± 1	± 6	± 7	16	15	9	± 6	± 1	± 7
5	7	11	± 2	± 4	± 6	16	14	10	± 4	± 2	± 6
6	7	12	± 1	± 5	± 6	15	14	9	± 5	± 1	± 6
6	8	11	± 2	± 3	± 5	15	13	10	± 3	± 2	± 5
7	8	11	± 1	± 3	± 4	14	13	10	± 3	± 1	± 4
8	9	11	± 1	± 2	± 3	13	12	10	± 2	± 1	± 3

Therefore, $\overline{SCF(19)} = \{2 3 17; 19 18 4; 2416; 19 17 5; \dots; 8 9 11, 13 12 10\}$.

Step 3: $STD(19) = SCF(19) \cup \overline{SCF(19)}$

$$= \{1 2 19, 13 18, 1 4 17, 15 16, 16 15, 1 7 14, 1 8 13, 1 9 12, 110 111\} \cup \{2 3 17, 19 18 4, 2 4 16, 19 17 5, \dots, 8 9 11, 13 12 10\}.$$

Step 4: Using the starter and addition modular 19 to enumerate $TD(19)$ as shown in table 5.

1	2	19	1	3	18	\dots	8	9	11	13	12	10
2	3	1	2	4	19	\dots	9	10	12	14	13	11
3	4	2	3	5	1	\dots	10	11	13	15	14	12
4	5	3	4	6	2	\dots	11	12	14	16	15	13
5	6	4	5	7	3	\dots	12	13	15	17	16	14
6	7	5	6	8	4	\dots	13	1	16	18	17	15
7	8	6	7	9	5	\dots	14	2	17	19	18	16
8	9	7	8	10	6	\dots	15	3	18	1	19	17
9	10	8	9	11	7	\dots	16	4	19	2	1	18
10	11	9	10	12	8	\dots	17	5	1	3	2	19
11	12	10	11	13	9	\dots	18	6	2	4	3	1
12	13	11	12	14	10	\dots	1	7	3	5	4	2
13	14	12	13	15	11	\dots	2	8	4	6	5	3
14	15	13	14	16	12	\dots	3	9	5	7	6	4
15	16	14	15	17	13	\dots	4	10	6	8	7	5
16	17	15	16	18	14	\dots	5	11	7	9	8	6
17	18	16	17	19	15	\dots	6	12	8	10	9	7
18	19	17	18	1	16	\dots	7	13	9	11	10	8
19	1	18	19	2	17	\dots	8	14	10	12	11	9

Table 5. $TD(19)$..

Our aim is to construct a new method for developing $TD(19)$. This method depends on building the starter of $TD(19)$ using interval techniques of the number of triples in the design.

4 INTERVALS CONSTRUCTIONS $TD(19)$

In this section, we use analyse and divide $STD(19)$ into intervals to construct formulas for $S_r TD(19)$, where $1 \leq r \leq 3$. From Step 3 of the previous section, we are able to summarize $STD(19)$ in terms of $S_r TD(19)$ and the triple number k as shown in Table 6.

<i>k</i>	1	2	...	9	10	11	...	22	23	24	25	...	32	33	34	35
$S_1TD(19)$	1	1	...	1	2	19	...	2	19	3	18	...	3	18	4	17
$S_2TD(19)$	2	3	...	10	3	18	...	9	12	4	17	...	8	13	5	16
$S_3TD(19)$	19	18	...	11	17	4	...	11	10	15	6	...	11	10	13	8

<i>k</i>	...	38	39	40	41	42	43	44	45	46	47	48	49	50	51
$S_1TD(19)$...	4	17	5	16	5	16	6	15	6	15	7	14	8	13
$S_2TD(19)$...	7	14	6	15	7	14	7	14	8	13	8	13	9	12
$S_3TD(19)$		11	10	12	9	11	10	12	9	11	10	11	10	11	10

Table 6. $S_rTD(19)$, where $1 \leq r \leq 3$

By Lemma 2.1, $1 \leq k \leq 51$. Let $[S_rTD(19)]$ be the k -th element in $S_rTD(19)$, for $1 \leq r \leq 3$. From Table 6, k can be divided into 7 intervals (periods). These intervals and the corresponding elements in $S_1TD(19)$ are illustrated in Table 7.

No. of Intervals	Intervals of k	Corresponding elements in $S_1TD(19)$
1	$1 \leq k \leq 9$	1, 1, 1, 1, 1, 1, 1, 1, 1
2	$10 \leq k \leq 23$	2, 19, 2, 19, 2, 19, 2, 19, 2, 19, 2, 19, 2, 19
3	$24 \leq k \leq 33$	3, 18, 3, 18, 3, 18, 3, 18, 3, 18
4	$34 \leq k \leq 39$	4, 17, 4, 17, 4, 17
5	$40 \leq k \leq 43$	5, 16, 5, 16
6	$44 \leq k \leq 47$	6, 15, 6, 15
7	$48 \leq k \leq 51$	7, 14, 8, 13

Table 7. Intervals of k and the corresponding elements in $S_1TD(19)$

It could be observed from Table 7 that the corresponding elements in $S_1TD(19)$ in the last interval of k need a special formula to produce them. Let f denotes the first number of the interval. The formula

$$y = \frac{1}{2} \left[k - f + \text{mod} \left(\frac{k+3}{2} \right) + 1 \right] \quad (1)$$

can be used to produce the corresponding elements in $S_1TD(19)$ in the indicated interval. From Tables 7, the construction of $S_1TD(19)$ is as follows.

$$[S_1TD(19)]_k = \begin{cases} 1 & 1 \leq k \leq 9 \\ 2 & 10 \leq k \leq 23, \text{ } k \text{ iseven} \\ 19 & 10 \leq k \leq 23, \text{ } k \text{ isodd} \\ 3 & 24 \leq k \leq 33, \text{ } k \text{ iseven} \\ 18 & 24 \leq k \leq 33, \text{ } k \text{ isodd} \\ 4 & 34 \leq k \leq 39, \text{ } k \text{ iseven} \\ 17 & \text{if } 34 \leq k \leq 39, \text{ } k \text{ isodd} \\ 5 & 40 \leq k \leq 43, \text{ } k \text{ iseven} \\ 16 & 40 \leq k \leq 43, \text{ } k \text{ isodd} \\ 6 & 44 \leq k \leq 47, \text{ } k \text{ iseven} \\ 15 & 44 \leq k \leq 47, \text{ } k \text{ isodd} \\ 6+y & 48 \leq k \leq 51, \text{ } k \text{ iseven} \\ 15-y & 48 \leq k \leq 51, \text{ } k \text{ isodd} \end{cases}$$

Similarly, intervals of k and the corresponding elements in $S_2TD(19)$ are shown in Table 8.

No. of Intervals	Intervals of k	Corresponding elements in $S_2TD(19)$
1	$1 \leq k \leq 9$	2, 3, 4, 5, 6, 7, 8, 9, 10
2	$10 \leq k \leq 23$	3, 18, 4, 17, 5, 16, 6, 15, 7, 14, 8, 13, 9, 12
3	$24 \leq k \leq 33$	4, 17, 5, 16, 6, 15, 7, 14, 8, 13
4	$34 \leq k \leq 39$	5, 16, 6, 15, 7, 14
5	$40 \leq k \leq 43$	6, 15, 7, 14
6	$44 \leq k \leq 47$	7, 14, 8, 13
7	$48 \leq k \leq 51$	8, 13, 9, 12

Table 8. Intervals of k and the corresponding elements in $S_2TD(19)$

Using formula (1) above to produce the corresponding elements in $S_2TD(19)$ in all intervals of k except the first one. Therefore, the construction of $S_2TD(19)$ is the following.

$$[S_2TD(19)]_k = \begin{cases} 1+k & 1 \leq k \leq 9 \\ 2+y & 10 \leq k \leq 23, \text{ } k \text{ iseven} \\ 19-y & 10 \leq k \leq 23, \text{ } k \text{ isodd} \\ 3+y & 24 \leq k \leq 33, \text{ } k \text{ iseven} \\ 18-y & 24 \leq k \leq 33, \text{ } k \text{ isodd} \\ 4+y & 34 \leq k \leq 39, \text{ } k \text{ iseven} \\ 17-y & \text{if } 34 \leq k \leq 39, \text{ } k \text{ isodd} \\ 5+y & 40 \leq k \leq 43, \text{ } k \text{ iseven} \\ 16-y & 40 \leq k \leq 43, \text{ } k \text{ isodd} \\ 6+y & 44 \leq k \leq 47, \text{ } k \text{ iseven} \\ 15-y & 44 \leq k \leq 47, \text{ } k \text{ isodd} \\ 7+y & 48 \leq k \leq 51, \text{ } k \text{ iseven} \\ 14-y & 48 \leq k \leq 51, \text{ } k \text{ isodd} \end{cases}$$

Finally, similar to the above discussion, intervals of k and the corresponding elements in $S_3TD(19)$ are shown in tables 9.

No. of Intervals	Intervals of k	Corresponding elements in $S_3TD(19)$
1	$1 \leq k \leq 9$	19, 18, 17, 16, 15, 14, 13, 12, 11
2	$10 \leq k \leq 23$	17, 4, 16, 5, 15, 6, 14, 7, 13, 8, 12, 9, 11, 10
3	$24 \leq k \leq 33$	15, 6, 14, 7, 13, 8, 12, 9, 11, 10
4	$34 \leq k \leq 39$	13, 8, 12, 9, 11, 10

5	$40 \leq k \leq 43$	12, 9, 11, 10
6	$44 \leq k \leq 47$	12, 9, 11, 10
7	$48 \leq k \leq 51$	11, 10, 11, 10

Table 9. Intervals of k and the corresponding elements in $S_3TD(19)$

Using the same formula (1), the construction of $S_2TD(19)$ is the following.

$$[S_3TD(19)]_k = \begin{cases} 20 - k & 1 \leq k \leq 9 \\ 18 - y & 10 \leq k \leq 23, k \text{ is even} \\ 3 + y & 10 \leq k \leq 23, k \text{ is odd} \\ 16 - y & 24 \leq k \leq 33, k \text{ is even} \\ 5 + y & 24 \leq k \leq 33, k \text{ is odd} \\ 14 - y & 34 \leq k \leq 39, k \text{ is even} \\ 7 + y \text{ if } & 34 \leq k \leq 39, k \text{ is odd} \\ 13 - y & 40 \leq k \leq 43, k \text{ is even} \\ 8 + y & 40 \leq k \leq 43, k \text{ is odd} \\ 13 - y & 44 \leq k \leq 47, k \text{ is even} \\ 8 + y & 44 \leq k \leq 47, k \text{ is odd} \\ 11 & 48 \leq k \leq 51, k \text{ is even} \\ 10 & 48 \leq k \leq 51, k \text{ is odd} \end{cases}$$

5 CONCLUSION

We have constructed a new method for developing the triad design of order 19, $TD(19)$. This method analyses the pattern of triples using interval techniques of the number of triples in the design. One can use this method to construct the general cases of $TD(v)$, where $v = 6n + 1$ or $v = 6n + 5$.

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References

- [1] Abu Saa, T. M., Ibrahim, H., Omer Z. & Rahim R. (2010). On particular enumeration for $TD(13)$. Proceeding of the 5th Social Economic & Information Technology (SEit), (pp. 33), Hatyai, Thailand.(23/11/2010-25/11/2010).
- [2] Ibrahim H. (2003). "A Family of Designs for Triads and Related Factorizations" PhD thesis.
- [3] Ibrahim, H., and Wallis, W. D., (2005). "An Enumeration of Triad Designs." J. Combinatorics Information and System Sciences: 5-17.
- [4] Ibrahim, H., Abu Saa, T. M., Omar Z. and Rahim, R., (2009.). "Difference set method and its application in developing some triad designs." Proceeding of the 5th Asian Mathematical Conference, Malaysia.
- [5] Mendelssohn, E., and A. R., and (1976). "One-factorization of the complete graph-A survey." Journal of Combinatorial Theory 20B: 265-282.
- [6] Schank, T., Wagner, D. (2005). Finding, Counting and Listing all Triangles in Large Graphs, An Experimental Study. WEA.
- [7] M. Latapy. Practical algorithms for triangle computations in very large (sparse (power-law)) graphs. Submitted, 2007.
- [8] Wallis, W. D., (1988). "Combinatorial Designs." Marcel Dekker, New York.
- [9] Wallis, W. D., (1997). "One-Factorizations." Kluwer, Boston.



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