An Enumeration of Triad Designs

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Abstract

In this paper we discuss a type of factorization called a *compatible factorization*. We introduce an invariant, the number of *k-fold edges*, and use it to enumerate isomorphism classes of these factorizations on seven points.

1 Introduction

We assume the standard ideas and definitions of graph theory. We denote the complete graph on n vertices by K_n.

A *one-factor* is a set of disjoint edges in a graph that together contain all the vertices, and a *one-factorization* of a graph G is a set of edge-disjoint one-factors that together contain all the edges of G. If G has an odd number of vertices, a *near-one-factor* consists of one vertex (the *focus*) and a set of disjoint edges that contain every other vertex, while a *near-one-factorization* is a set of edge-disjoint near-one-factors that together contain all the edges. Given the near-one-factor

$$N = x \;\; ab \;\; cd \;\; \ldots \;\; yz$$

it will be convenient to refer to

$$\{xab\} \;\; \{xcd\} \;\; \ldots \;\; \{xyz\}$$

as the (set of) *triples associated with* N. A comprehensive discussion and bibliography of the literature concerning one-factorizations can be found in [7], [5] and [8].

The paper [9] introduces some designs whose blocks are ordered triples, or *triads*, subject to certain restrictions. We wish to discuss in detail one of the designs in that paper, namely design B. This is a design with 7 treatments, whose blocks are the 35 triples on the treatments. The blocks are arranged in seven rounds, such that:

(i) each round contains the same number of blocks;

(ii) each block occurs exactly once in the design;

(iii) each treatment occurs either twice or three times in each round;

(iv) no two treatments occurs together in two or more blocks in any round.

We shall refer to a design satisfying (i) through (iv) as a *triad design* on 7 points. (In the original application, it was required that the blocks be ordered so that no treatment occurs twice in the same position in any round. However, this condition can obviously be met by any triad design.) It follows immediately that each round of such a design contains five blocks; one treatment occurs in three of the blocks and the others in two each. If we refer to the treatment of frequency 3 as the *focus* of the round, then each treatment is the focus of exactly one round. It is convenient to label the treatments as 1, 2, 3, 4, 5, 6, 7, and to order the rounds so that treatment i is the focus of round i. From property (iv), round i is of the form

$$ix_1^iy_1^i,\ \ ix_2^iy_2^i,\ \ ix_3^iy_3^i,\ \ x_1^ix_2^ix_3^i,\ \ y_1^iy_2^iy_3^i,$$

where $\{i, x_1^i, y_1^i, x_2^i, y_2^i, x_3^i, y_3^i\}$ is a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$. The first three triples in round i are the triples associated with the near-one-factor

$$N_i = \{i\ \ x_1^iy_1^i\ \ x_2^iy_2^i\ \ x_3^iy_3^i\}.$$

Table 1 shows the example given in [9]. This is a highly structured example. The seven near-one-factors N_i associated with the rounds form

127	316	451	532	674
231	427	562	643	715
342	531	673	754	126
453	642	714	165	237
564	753	125	276	341
675	164	236	317	452
712	275	347	321	563

Table 1: A design for seven treatments

the near-one-factorization

1	27	36	45
2	31	47	56
3	42	51	67
4	53	62	71
5	64	73	12
6	75	14	23
7	12	25	34

but that is not necessary for the design.

The seven near-one-factors must satisfy the conditions that each treatment occurs as a focus exactly once and the totality of triples associated with them contains no repetition. To discuss this we define such a set of near-one-factors to be a *compatible factorization* or CF. This name derives from the following consideration: if we form a multigraph whose vertices are the treatments, and whose edges are the union (with multiplicities preserved) of the factors in a CF, then the factors can be viewed as a factorization of this multigraph. We shall call the multigraph the *graph of* the compatible factorization.

It is natural to ask whether there exist compatible factorizations more general than near-one-factorizations, and how common they are. In the following sections we formally define compatible factorizations and examine the number of non-isomorphic examples for the case of 7 treatments. The number of possibilities is surprisingly large.

2 Preliminaries and Definitions

Consider the following combinatorial problem: we are given v vertices and we wish to construct an array from a certain set of triples of these vertices; each triple must occur at most once in the array. We must arrange the elements in rows and columns such that in each row no pair of vertices is repeated and the number of rows equals the number of given vertices. For example say $v = 7$ and we want to produce the following triples:

$$\begin{array}{lllllll} \{123\}, & \{124\}, & \{136\}, & \{134\}, & \{125\}, & \{126\}, & \{137\}, \\ \{145\}, & \{167\}, & \{236\}, & \{235\}, & \{245\}, & \{345\}, & \{367\}, \\ \{267\}, & \{257\}, & \{347\}, & \{467\}, & \{567\}, & \{456\}, & \{457\} \end{array}$$

We have 21 different triples with seven vertices and each vertex appears precisely nine times, but each pair of elements does not appear a constant number of times. Since we have seven elements, we shall have seven rows, and consequently three columns. One construction is:

$$\begin{array}{rcccc} F_1: & 1 & 23 & 45 & 67 \\ F_2: & 2 & 14 & 36 & 57 \\ F_3: & 3 & 16 & 25 & 47 \\ F_4: & 4 & 13 & 25 & 67 \\ F_5: & 5 & 12 & 34 & 67 \\ F_6: & 6 & 12 & 37 & 45 \\ F_7: & 7 & 13 & 26 & 45 \\ & C_1 & C_2 & C_3 & C_4 \end{array}$$

In this case we have 7 rows and 4 columns. Now in order to form a triple, we append C_1 with C_2, C_1 with C_3, and C_1 with C_4, obtaining 3 triples in each row. For instance in F_1 we have triples {123},{145} and {167}; continuing in the same fashion in F_2 through F_7 we produce the desired triples. Any solution to this problem will be called a *compatible factorization* of order v and will be denoted by $CF(v)$.

Definition. A compatible factorization of order v, or $CF(v)$, is a $v \times \frac{v-1}{2}$ array that satisfies the following conditions:

(i) The entries in row i form a near-one-factor with focus i.

(ii) The triples associated with the rows contain no repetitions.

Note that the triples are unordered. For example $\{456\}$ and $\{546\}$ are considered the same triple.

An obvious necessary condition for the existence of a $CF(v)$ is that v must be odd.

Theorem 2.1. *There exists a compatible factorization for every odd order* $v > 3$*.*

Proof. Suppose $v = 2t + 1 > 3$. The near-one-factor from the patterned starter, with i-th factor

$$i \;\; (i+1)(i-1) \;\; (i+2)(i-2) \;\; \ldots \;\; (i+n)(i-n) \quad \bmod v,$$

is a compatible factorization. □

No $CF(3)$ can exist: with the three symbols $1, 2, 3$ the only possible near-one-factor with focus 1 is 1 23, the only possible near-one-factor with focus 2 is 2 13, and these two have a common associated triple.

3 Isomorphism classes of compatible factorizations

We wish to discuss variability of triad designs. It is clear that isomorphic triad designs have isomorphic compatible factorizations. So we wish to discuss isomorphism classes of compatible factorizations. It is easy to see there is a unique $CF(5)$ up to isomorphism. In order to discuss the case $v = 7$ we introduce some definitions.

Definition. A compatible factorization has a *k-fold edge* if there is an edge common to k of the factors.

Definition. The *2-factor intersection* or *2-fi* of a $CF(7)$ is the graph whose vertices are the factors, where two vertices are joined by an edge when the two factors have a common edge.

Theorem 3.1. *There are no* 4*-fold edges or* 5*-fold edges in compatible factorizations of order seven.*

Proof. Suppose (67) is a 4-fold edge in a $CF(7)$. The factors in the compatible factorization must look like

1	--	--	67
2	--	--	67
3	--	--	67
4	--	--	67
5	--	--	--
6	--	--	$x7$
7	--	--	$y6$

for some x and y in $\{1,2,3,4,5\}$. Observe that x cannot equal 1, for then factor 6 would contain triple 167 which already occurred in factor 1. Similarly x cannot be 2, 3, or 4 because triples 267, 367 or 467 would appear twice. So $x = 5$. If we consider factor 7, a similar argument shows that $y = 1,2,3,$ or 4 is impossible. However, if $y = 5$, then 567 appears twice. So the factorization cannot be completed, and a 4-fold edge is impossible. As a 5-fold edge is also 4-fold, this proof also shows there can be no 5-fold edge. □

By theorem 3.1, we can categorize an analysis of isomorphism classes of $CF(7)$s into three parts: CFs with 3-fold edges, CFs with 2-fold edges but no 3-fold edges, and CFs without multifold edges. We shall outline the case of 3-fold edges, and state the results for the other two cases; details of those are left to the reader, or can be found in [4].

4 The case of 3-fold edges

Lemma 4.1. *There are exactly sixty-five isomorphism classes of compatible factorizations of order seven that contain* 3*-fold edges.*

Proof. Consider a compatible factorization of order seven with vertices $\{1,2,3,4,5,6,7\}$. We denote the factors by A,B,C,D,E,F,G and take them in that order (so that 1 is the isolate in A, 2 is the isolate in B, 3 is the isolate in C, etc.). Without loss of generality we can assume that the 3-fold set is $\{A,B,C\}$ and write

$$\begin{array}{lcllll} \text{A} & = & 1 & 23 & 45 & 67 \\ \text{B} & = & 2 & 14 & 35 & 67 \\ \text{C} & = & 3 & 15 & 24 & 67. \end{array}$$

There are exactly six possible fourth factors:

$$\begin{array}{lcllll} D_1 & = & 4 & 13 & 26 & 57 \\ D_2 & = & 4 & 13 & 27 & 56 \\ D_3 & = & 4 & 16 & 25 & 37 \end{array} \qquad \begin{array}{lcllll} D_4 & = & 4 & 16 & 27 & 35 \\ D_5 & = & 4 & 17 & 25 & 36 \\ D_6 & = & 4 & 17 & 26 & 35 \end{array}$$

We can eliminate three of these cases from the consideration of isomorphism classes by carrying out the permutation (67):

$$\begin{array}{rcl} (A, B, C, D_1)(67) & = & (A, B, C, D_2) \\ (A, B, C, D_3)(67) & = & (A, B, C, D_5) \\ (A, B, C, D_4)(67) & = & (A, B, C, D_6) \end{array}$$

So we can complete the set of possible compatible factorizations that start $\{A, B, C\}$ by considering factors D_1, D_3 and D_4.

Case $D = D_1$.

The possible candidates for E, F and G are:

$$\begin{array}{lcl} E_1 & = & 5\ 12\ 37\ 46 \\ E_2 & = & 5\ 16\ 24\ 37 \\ E_3 & = & 5\ 16\ 27\ 34 \\ E_4 & = & 5\ 17\ 24\ 36 \\ E_5 & = & 5\ 17\ 26\ 34 \\ & & \end{array} \qquad \begin{array}{lcl} F_1 & = & 6\ 12\ 34\ 57 \\ F_2 & = & 6\ 12\ 35\ 47 \\ F_3 & = & 6\ 13\ 25\ 47 \\ F_4 & = & 6\ 15\ 23\ 57 \\ F_5 & = & 6\ 15\ 23\ 47 \\ & & \end{array} \qquad \begin{array}{lcl} G_1 & = & 7\ 12\ 34\ 56 \\ G_2 & = & 7\ 12\ 35\ 46 \\ G_3 & = & 7\ 13\ 24\ 56 \\ G_4 & = & 7\ 13\ 25\ 46 \\ G_5 & = & 7\ 14\ 23\ 56 \\ G_6 & = & 7\ 15\ 23\ 46 \end{array}$$

From these possible factors, we can generate fifty-three different combinations of $CF(7)$ with a 3-fold edge as shown in Table 2.

As a first step in classifying these compatible factorizations, we calculate the 2-factor intersection of each factorization. The size (number of edges) of this graph is shown in Table 2 after the name of the factorization, in parentheses. If the graphs of two factorizations have different sizes, the factorizations will certainly be non-isomorphic. If they are the same size, but not isomorphic, the factorizations are again non-isomorphic. If the graphs

5 12 37 46 6 12 34 57 7 13 25 46 $\mathcal{F}_1(7)$	5 12 37 46 6 12 34 57 7 15 23 46 $\mathcal{F}_2(8)$	5 12 37 46 6 12 35 47 7 12 34 56 $\mathcal{F}_3(7)$	5 12 37 46 6 12 35 47 7 13 24 56 $\mathcal{F}_4(7)$	5 12 37 46 6 12 35 47 7 14 23 56 $\mathcal{F}_5(7)$	5 12 37 46 6 13 25 47 7 12 34 56 $\mathcal{F}_6(5)$
5 12 37 46 6 13 25 47 7 13 24 56 $\mathcal{F}_{3a}(7)$	5 12 37 46 6 13 25 47 7 14 23 56 $\mathcal{F}_7(6)$	5 12 37 46 6 14 23 57 7 13 25 46 $\mathcal{F}_{2a}(8)$	5 12 37 46 6 14 23 57 7 15 23 46 $\mathcal{F}_8(9)$	5 12 37 46 6 15 23 47 7 12 34 56 $\mathcal{F}_{7a}(6)$	5 12 37 46 6 15 23 47 7 13 24 56 $\mathcal{F}_{5a}(7)$
5 12 37 46 6 15 23 47 7 14 23 56 $\mathcal{F}_9(8)$	5 16 24 37 6 12 34 57 7 13 25 46 $\mathcal{F}_{10}(6)$	5 16 24 37 6 12 34 57 7 15 23 46 $\mathcal{F}_{11}(7)$	5 16 24 37 6 12 35 47 7 12 34 56 $\mathcal{F}_{12}(6)$	5 16 24 37 6 12 35 47 7 13 24 56 $\mathcal{F}_{13}(8)$	5 16 24 37 6 12 35 47 7 14 23 56 $\mathcal{F}_{14}(7)$
5 16 24 37 6 13 25 47 7 12 34 56 $\mathcal{F}_{15}(5)$	5 16 24 37 6 13 25 47 7 13 24 56 $\mathcal{F}_{16}(9)$	5 16 24 37 6 13 25 47 7 14 23 56 $\mathcal{F}_{17}(7)$	5 16 24 37 6 14 23 57 7 13 25 46 $\mathcal{F}_{18}(8)$	5 16 24 37 6 14 23 57 7 15 23 46 $\mathcal{F}_{19}(10)$	5 16 27 34 6 12 34 57 7 12 35 46 $\mathcal{F}_{20}(7)$
5 16 27 34 6 12 34 57 7 15 23 46 $\mathcal{F}_{21}(7)$	5 16 27 34 6 12 35 47 7 12 34 56 $\mathcal{F}_{22}(6)$	5 16 27 34 6 12 35 47 7 13 24 56 $\mathcal{F}_{23}(6)$	5 16 27 34 6 12 35 47 7 14 23 56 $\mathcal{F}_{24}(6)$	5 16 27 34 6 13 25 47 7 12 34 56 $\mathcal{F}_{25}(5)$	5 16 27 34 6 13 25 47 7 13 24 56 $\mathcal{F}_{26}(7)$
5 16 27 34 6 13 25 47 7 14 23 56 $\mathcal{F}_{27}(6)$	5 16 27 34 6 14 23 57 7 12 35 46 $\mathcal{F}_{28}(7)$	5 16 27 34 6 14 23 57 7 15 23 46 $\mathcal{F}_{29}(9)$	5 17 24 36 6 12 34 57 7 12 35 46 $\mathcal{F}_{30}(7)$	5 17 24 36 6 12 34 57 7 13 25 46 $\mathcal{F}_{31}(6)$	5 17 24 36 6 13 25 47 7 12 34 56 $\mathcal{F}_{32}(6)$
5 17 24 36 6 13 25 47 7 13 24 56 $\mathcal{F}_{33}(9)$	5 17 24 36 6 13 25 47 7 14 23 56 $\mathcal{F}_{34}(7)$	5 17 24 36 6 14 23 57 7 12 35 46 $\mathcal{F}_{35}(10)$	5 17 24 36 6 14 23 57 7 13 25 46 $\mathcal{F}_{36}(8)$	5 17 24 36 6 15 23 47 7 12 34 56 $\mathcal{F}_{37}(6)$	5 17 24 36 6 15 23 47 7 13 24 56 $\mathcal{F}_{38}(9)$
5 17 24 36 6 15 23 47 7 14 23 56 $\mathcal{F}_{29a}(9)$	5 17 26 34 6 12 34 57 7 12 35 46 $\mathcal{F}_{39}(8)$	5 17 26 34 6 12 34 57 7 13 25 46 $\mathcal{F}_{40}(7)$	5 17 26 34 6 12 35 47 7 12 34 56 $\mathcal{F}_{41}(7)$	5 17 26 34 6 12 35 47 7 13 24 56 $\mathcal{F}_{42}(7)$	5 17 26 34 6 12 35 47 7 14 23 56 $\mathcal{F}_{43}(7)$
5 17 26 34 6 14 23 57 7 12 35 46 $\mathcal{F}_{44}(8)$	5 17 26 34 6 14 23 57 7 13 25 46 $\mathcal{F}_{45}(8)$	5 17 26 34 6 15 23 47 7 12 34 56 $\mathcal{F}_{46}(7)$	5 17 26 34 6 15 23 47 7 13 24 56 $\mathcal{F}_{47}(8)$	5 17 26 34 6 15 23 47 7 14 23 56 $\mathcal{F}_{48}(9)$	

Table 2: Possible CF when $D = D_1$

are isomorphic, this often helps to prove the factorizations are isomorphic. We illustrate by looking at cases with size 5 *2-fi* and size 6 *2-fi*.

Size 5 2-factor intersection.

There are precisely three factorizations with size 5 *2-fi*, namely $\mathcal{F}_6$, $\mathcal{F}_{15}$, and $\mathcal{F}_{25}$. The graphs are shown in Figure 1.

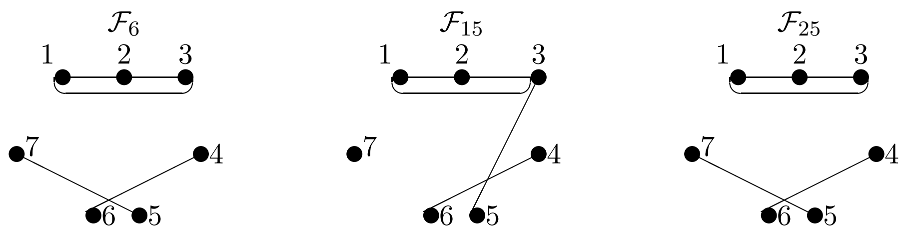

Figure 1: The size 5 *2-fi*

The graphs of $\mathcal{F}_6$ and $\mathcal{F}_{25}$ are isomorphic, so we test whether the factorizations isomorphic or not. It is easy to see in this case that they are not isomorphic. An isomorphism from $\mathcal{F}_6$ to $\mathcal{F}_{25}$ must carry $\{1,2,3\}$ to $\{1,2,3\}$ and $\{\{4,6\},\{5,7\}\}$ to $\{\{4,6\},\{5,7\}\}$. As 123 is a triple in factor A, any isomorphism must map factor A of $\mathcal{F}_6$ to factor A of $\mathcal{F}_{25}$, so $1 \to 1$. If $2 \to 2$ and $3 \to 3$ the only possibility is the identity map, and if $2 \leftrightarrow 3$ the only possibility is (23)(45)(67), neither of which maps $\mathcal{F}_6$ to $\mathcal{F}_{25}$.

Size 6 2-factor intersection.

There are exactly eleven compatible factorizations with *2-fi* of size 6; their graphs are shown in Figure 2.

The graphs of $\mathcal{F}_7$, $\mathcal{F}_{7a}$, $\mathcal{F}_{27}$, and $\mathcal{F}_{32}$ are isomorphic so we consider these factorizations together. For the same reason we consider $\mathcal{F}_{10}$ together with $\mathcal{F}_{31}$, $\mathcal{F}_{12}$ with $\mathcal{F}_{23}$ and $\mathcal{F}_{24}$ with $\mathcal{F}_{37}$; $\mathcal{F}_{22}$ is not isomorphic to any of the others.

First consider $\mathcal{F}_7$ and $\mathcal{F}_{7a}$. There are exactly four permutations that carry the *2-fi* of $\mathcal{F}_7$ to that of $\mathcal{F}_{7a}$. They are (23)(45)(67), (23)(4765), (132)(45)(67), and (132)(4765). On testing we find that $\mathcal{F}_7(23)(45)(67) = \mathcal{F}_{7a}$, so $\mathcal{F}_7$ is isomorphic to $\mathcal{F}_{7a}$.

Next consider $\mathcal{F}_7$ and $\mathcal{F}_{27}$. The graphs look exactly the same, but the compatible factorizations are not isomorphic, There are eight possible permutations, of the form $(12)^\alpha(37)^\beta(46)^\gamma$ (where α, β, γ can be 0 or 1), and

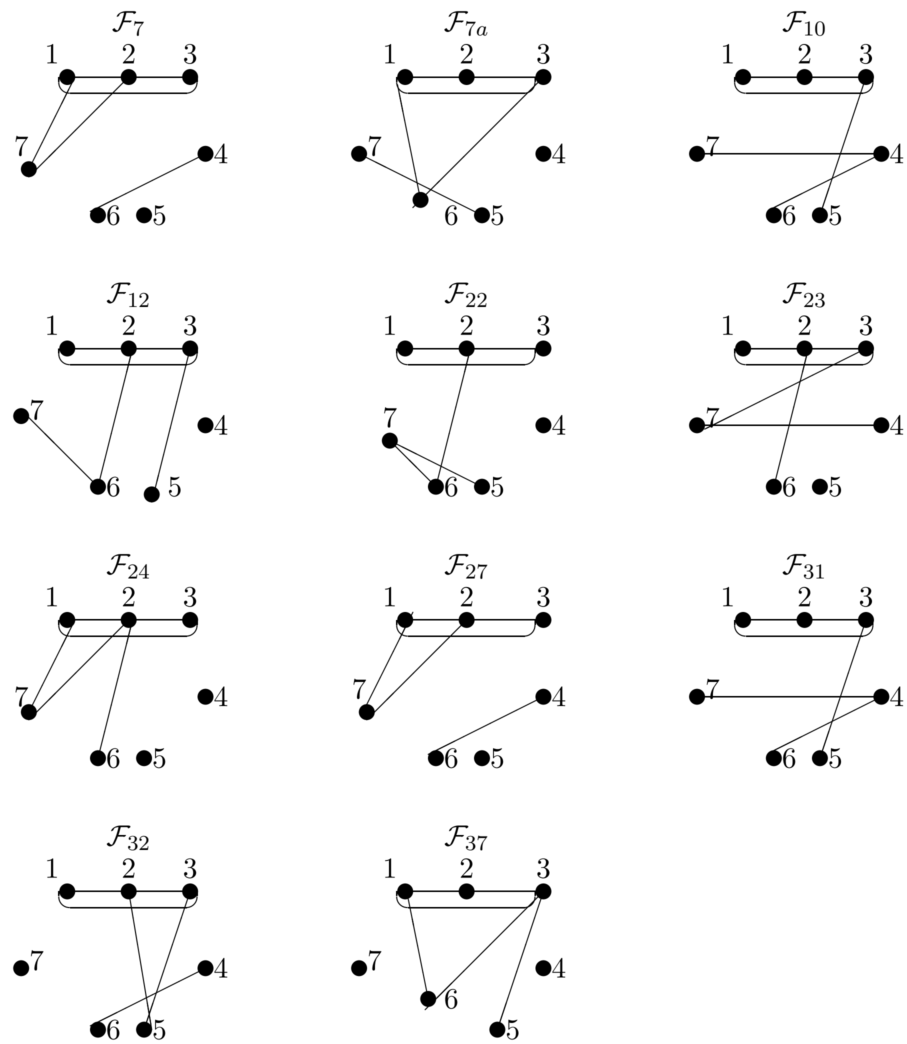

Figure 2: Size 6 *2-fi*

none is an isomorphism.

Now consider $\mathcal{F}_7$ and $\mathcal{F}_{32}$. Any isomorphism from $\mathcal{F}_7$ to $\mathcal{F}_{32}$ must map 1 to 2 or 3. If 1 maps to 2, then 2 must map to 3, and factor 2 cannot map correctly. If 1 maps to 3, then 2 must map to 5, which is impossible. Thus they are not isomorphic. A similar argument shows that $\mathcal{F}_{31}$ is not isomorphic to $\mathcal{F}_{36}$. We can also show that $\mathcal{F}_{10}$ and $\mathcal{F}_{31}$ are different, as are $\mathcal{F}_{12}$ and $\mathcal{F}_{23}$, and also $\mathcal{F}_{24}$ and $\mathcal{F}_{37}$.

From this discussion we see that only $\mathcal{F}_7$ and $\mathcal{F}_{7a}$ are isomorphic. We carried out similar computations for all *2-fi* of size 7, size 8, size 9 and size 10. We found that for size 7, $\mathcal{F}_3$ is isomorphic to $\mathcal{F}_{3a}$ and $\mathcal{F}_5$ is isomorphic to $\mathcal{F}_{5a}$; for size 8, $\mathcal{F}_2$ is isomorphic to $\mathcal{F}_{2a}$; and for size 9, $\mathcal{F}_{29}$ is isomorphic to $\mathcal{F}_{29a}$. Thus for D_1, we only have forty-eight cases left. It will be observed that we cunningly labeled Table 2 so that those cases numbered with numbers alone — no letters appended — form a representative set.

Case $D = D_3$.

We have thirty-six CF; after classifing up to isomorphism we have nine CF left as listed in table 3.

5 16 27 34	5 16 27 34	5 16 27 34	5 16 27 34	5 17 26 34
6 12 34 57	6 12 34 57	6 12 35 47	6 12 35 47	6 12 34 57
7 12 35 46	7 15 23 46	7 13 24 56	7 14 23 56	7 12 35 46
$\mathcal{F}_{49}$	$\mathcal{F}_{50}$	$\mathcal{F}_{51}$	$\mathcal{F}_{52}$	$\mathcal{F}_{53}$
5 17 26 34	5 17 26 34	5 17 26 34	5 17 26 34	
6 12 34 57	6 12 35 47	6 12 35 47	6 13 24 57	
7 13 25 46	7 13 24 56	7 14 23 56	7 12 35 46	
$\mathcal{F}_{54}$	$\mathcal{F}_{55}$	$\mathcal{F}_{56}$	$\mathcal{F}_{57}$	

Table 3: Possible CF when $D = D_3$

Case $D = D_4$.

In this case if we enumerate all the possible cases, we also find that we have thirty-six possible combinations. Using a similar argument we reduce to eight different compatible factorizations as displayed in table 4.

So there are sixty-five nonisomorphic classes of $CF(7)$ with 3-fold edges. □

5 The case of 2-fold edges without 3-fold edges

Lemma 5.1. *There are exactly 164 compatible factorizations which contain a* 2-fold-edge *but no* 3-fold-edge*.*

Proof. Suppose factors A and B form a 2-fold edge. Without loss of gener-

5 16 24 37	5 16 24 37	5 16 24 37	5 16 24 37
6 12 34 57	6 12 34 57	6 12 35 47	6 12 35 47
7 13 25 46	7 15 23 46	7 12 34 56	7 14 23 56
$\mathcal{F}_{58}$	$\mathcal{F}_{59}$	$\mathcal{F}_{60}$	$\mathcal{F}_{61}$
5 17 24 36	5 17 24 36	5 17 24 36	5 17 24 36
6 12 34 57	6 12 34 57	6 13 25 47	6 13 25 47
7 12 35 46	7 13 25 46	7 12 34 56	7 14 23 56
$\mathcal{F}_{62}$	$\mathcal{F}_{63}$	$\mathcal{F}_{64}$	$\mathcal{F}_{65}$

Table 4: Possible CF when $D = D_4$

ality we can take

$$\begin{array}{lcllll} \text{A} & = & 1 & 23 & 45 & 67 \\ \text{B} & = & 2 & 14 & 35 & 67 \end{array}$$

Now factor C cannot have edge (67) otherwise A, B, C will contain a 3-fold edge. There are six possible factors C:

$$\begin{array}{lcllll} C_1 & = & 3 & 14 & 26 & 57 \\ C_2 & = & 3 & 14 & 27 & 56 \\ C_3 & = & 3 & 15 & 26 & 47 \\ C_4 & = & 3 & 15 & 37 & 46 \end{array} \qquad \begin{array}{lcllll} C_5 & = & 3 & 16 & 24 & 57 \\ C_6 & = & 3 & 16 & 27 & 45 \\ C_7 & = & 3 & 17 & 24 & 56 \\ C_8 & = & 3 & 17 & 26 & 45. \end{array}$$

The permutation (67) leaves factor A and B fixed and reduce factor C to only four possible combinations, namely C_1, C_3, C_5 and C_6.

We now proceed as before (for details, see [4]). There are 164 isomorphism classes, labeled F_{66} to F_{229}; to save space, these are listed online at [10]. □

6 The case of no multiple edges

Lemma 6.1. *There are exactly two compatible factorizations of order seven with no multiple edge.*

Proof. A compatible factorization with no multiple edges is a near-one-factor for which the totality of associated triples contains no repetitions. It is easy to see that there are exactly six isomorphism classes of near-one-factors on seven points; of these, there are two have no repeated triples. They are shown in Table 5. □

1	23	45	67	1	23	45	67
2	14	36	57	2	14	36	57
3	16	25	47	3	15	27	46
4	17	26	35	4	17	26	35
5	12	37	46	5	16	24	37
6	15	27	34	6	13	25	47
7	13	24	56	7	12	34	56
		F_{230}			F_{231}		

Table 5: All CF with no multiple edge

7 Triad designs

It is now a simple matter to test whether each compatible factorization can be embedded in a triad design. A complete search shows that precisely six of the designs can be embedded. Not surprisingly, each factorization can be embedded in exactly one way. (It is not inconceivable that the unused triples could be allocated in more than one way, but this does not occur in the case of seven symbols.) So there are precisely six triad designs on seven treatments, up to isomorphism. The six triad designs are shown in Table 6. Each is labeled with the name of the corresponding compatible factorization.

8 Summary

From lemmas 4.1, 5.1 and 6.1 we have:

Theorem 8.1. *There are precisely 231 nonisomorphic compatible-factorizations of order seven.*

From Section 7 we have:

123	145	167	257	346	123	145	167	256	347
214	235	267	156	347	214	235	267	157	346
315	324	367	147	256	315	324	367	146	257
413	426	457	127	356	413	426	457	127	356
512	537	546	136	247	512	537	546	136	247
614	623	657	137	245	615	623	647	137	245
715	723	746	126	345	714	723	756	126	345
		$\mathcal{F}_8$					$\mathcal{F}_9$		
123	145	167	247	356	123	145	167	256	347
214	235	267	156	347	214	235	267	136	457
314	326	357	125	467	314	326	357	127	456
417	423	456	136	257	417	426	435	125	367
517	526	534	146	237	517	524	536	146	237
612	634	657	137	245	612	634	657	135	247
712	736	745	246	135	713	725	746	156	234
		$\mathcal{F}_{154}$					$\mathcal{F}_{183}$		
123	145	167	247	356	123	145	167	356	247
214	235	267	137	456	214	236	257	137	456
315	326	347	146	257	315	327	346	126	457
413	426	457	125	367	417	426	435	125	367
516	524	537	127	346	516	524	537	134	267
613	625	647	157	234	613	625	647	157	234
714	723	756	126	345	712	734	756	146	235
		$\mathcal{F}_{189}$					$\mathcal{F}_{231}$		

Table 6: All triad designs on seven treatments

Theorem 8.2. *There are precisely six nonisomorphic triad designs of order seven.*

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