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Hermite-Hadamard inequality for product of (h_1, h_2, s)-convex and m-harmonically convex function

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Abstract

In this paper, a new definition of (m,h_1,h_2,s)-Harmonically convex function is introduced by combining *m*-convex, (h_1,h_2)-convex, *s*-convex, and harmonically convex function. Nowadays the approach of combining different convex functions is being used to extend the mathematical inequalities. In this paper, H-H inequality is considered to extend the fact that the combination of two or more convex functions combines their properties also. This innovative approach of combining convex functions leads to new applications in a variety of domains, including mathematics as well as other fields. These given inequalities can be considered as refinements and improvements to previously established results.

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1. Introduction

Convex functions are very important because it provides the basis for construction of mathematical inequalities.

A function $f:\mathbb{I}\to\mathbb{R},$ is convex for $a_1\in[0,1],$ if

$$f(a_1x+m(1-a_1)y)\leq a_1f(x)+(1-a_1)f(y),\ x,y\in\mathbb{I}. \tag{1}$$

Several mathematical inequalities and extensions are the results of convex functions [1-9]. The first inequality in literature for convex function is Hermite-Hadamard inequality [7].

Let $f:\mathbb{I}\to\mathbb{R}_+$ *be a convex function, then*

$$f\left[\frac{p+q}{2}\right]\leq\frac{1}{q-p}\int_p^q f(x)dx\leq\frac{f(q)+f(p)}{2},\ p,q\in\mathbb{I}\in\mathbb{R}, \tag{2}$$

is called Hermite-Hadamard (H-H) inequality.

Many researchers used to extend H-H inequality using different convex function [13-15]. The new trend is to combine more than two different convex function and to extend this inequality.

In [10], a new definition of (m,h_1,h_1)-convex function is introduced as

Definition 1: A function $f:\mathbb{I}\to(0,\infty)$ and $h_1,h_2:\mathbb{I}\to\mathbb{R},$ then f is called (m,h_1,h_2)-convex for $a_1\in[0,1],$ if

$$f(a_1x+m(1-a_1)y)\leq h_1(a_1)f(x)+mh_2(1-a_1)f(y),\quad x,y\in\mathbb{I}. \tag{3}$$

Here, $h\neq 0$ is a positive function.

Recently, the definition of harmonically convex function is introduced by Iscan [11] as follows.

Definition 2: A function $f : \mathbb{I} \to \mathbb{R}$, is called harmonically convex on $\mathbb{I} \subset \mathbb{R} \setminus \{0\}$, if

$$f\left(\frac{xy}{a_1 x + m(1-a_1)y}\right) \leq a_1 f(x) + (1-a_1) f(y),\ x, y \in \mathbb{I}, a_1 \in [0,1] \tag{4}$$

On the basis of this definition (2), some new extensions for H-H inequality are produced as

Theorem 1: *Let* $f : \mathbb{I} \to \mathbb{R}$ *be a harmonically convex function on* $\mathbb{I} \subset \mathbb{R} \setminus \{0\}$, *then the following H-H inequality holds.*

$$f\left[\frac{2pq}{p+q}\right] \leq \frac{pq}{q-p} \int_p^q \frac{f(x)}{x^2} dx \leq \frac{f(q) + f(p)}{2},\ p, q \in \mathbb{I} \in \mathbb{R}, \tag{5}$$

In [12], Xi and Zhang introduced another definition by combining *m*-convex and harmonically convex function as follows.

Definition 3: A function $f : \mathbb{I} \to \mathbb{R}$, is called m-harmonically convex on $\mathbb{I} \subset \mathbb{R} \setminus \{0\}$, if

$$f\left(\frac{xy}{a_1 x + m(1-a_1)y}\right) \leq a_1 f(x) + m(1-a_1) f(y),\ \ x, y \in \mathbb{I}, a_1 \in [0,1] \tag{6}$$

Here $m \in (0,1]$ is a constant quantity.

The H-H inequality for *m*-harmonically convex function was also investigated in [12]. Furthermore, two more theorems are also discussed in that study as follows.

Theorem 2: *Let* $f : \mathbb{I} \to \mathbb{R}$ *be a m-harmonically convex function on* $\mathbb{I} \subset \mathbb{R} \setminus \{0\}$ *and* $\mathbb{I} \in [0,1]$, *then*

$$\left[\frac{pq}{q-p}\right] \int_p^q \frac{f(x)}{x^2} dx \leq \min\{f(q) + mf(mp), f(p) + mf(mq)\} \int_0^1 h(a_1) da_1. \tag{7}$$

Theorem 3: *Let* $f : \mathbb{I} \to \mathbb{R}$ *be a m-harmonically convex function on* $\mathbb{I} \subset \mathbb{R} \setminus \{0\}$ *and* $\mathbb{I} \in [0,1]$, *then*

$$\frac{1}{h\left(\frac{1}{2}\right)} f\left[\frac{2pq}{q-p}\right] \leq \frac{pq}{q-p}\int_p^q \frac{f(x)+mf(xm)}{x^2}dx \leq \frac{1}{2}\{f(p)+f(q)$$
$$+2m[f(pm)+f(qm)]+m^2[f(am^2)+f(bm^2)]\}\int_0^1 h(a_1)da_1. \quad (8)$$

Theorem 4: *Let* $f, g: \mathbb{I} \to \mathbb{R}$ *be two m-harmonically convex functions named m_1-harmonically convex and m_2-harmonically convex function respectively on* $\mathbb{I} \subset \mathbb{R} \setminus \{0\}$ *and* $\mathbb{I} \in [0,1]$, *then*

$$\frac{pq}{q-p}\int_p^q \frac{f(x)g(x)}{x^2}dx \leq min\{M_1(x,y), M_2(x,y)\}. \quad (9)$$

Here,

$$M_1(x,y) = [m_1 f(pm_1)m_2 g(pm_2)+f(q)g(q)]\int_0^1 [h(a_1)]^2 da_1 +$$
$$[m_1 f(pm_1)g(q)+m_2 g(pm_2)f(q)]\int_0^1 h(a_1)h(1-a_1)da_1$$
$$M_1(x,y) = [m_1 f(qm_1)m_2 g(pm_2)+f(p)g(p)]\int_0^1 [h(a_1)]^2 da_1 +$$
$$[m_1 f(pm_1)g(q)+m_2 g(pm_2)f(q)]\int_0^1 h(a_1)h(1-a_1)da_1$$

2. Main Results

In this section, we introduced a new definition of (m, h_1, h_2, s)-HA-convex function as follows

Definition 4: A function $f: \mathbb{I} \to \mathbb{R}$, is called (m, h_1, h_2, s)-HA convex on $\mathbb{I} \subset \mathbb{R} \setminus \{0\}$, if

$$f\left(\frac{xy}{a_1 x + m(1-a_1)y}\right) \leq h_1^s(a_1)f(x) + mh_2^s(1-a_1)f(y), \quad x, y \in \mathbb{I}, a_1 \in [0,1]. \quad (10)$$

Here $m \in (0,1]$ is a constant quantity and $h_1, h_2 \neq 0$ for $h_1, h_2 : [0,1] \to \mathbb{R}$.

Remark:

(1) If $h_1^s(a_1) = h^s(a_1)$ and $h_2^s(1-a_1) = h^s(1-a_1)$ then this Definition (4) will be reduced to (m, h, s)-HA convex function.

(2) If $h_1^s(a_1) = h(a_1)$ and $h_2^s(1-a_1) = h(1-a_1)$ then this Definition (4) will be reduced to (m,h)-HA convex function.

(3) If $h_1^s(a_1) = a_1$ and $h_2^s(1-a_1) = (1-a_1)$ then this Definition (4) will be reduced to m-HA convex function.

(4) If $h_1^s(a_1) = a_1^{\,s}$ and $h_2^s(1-a_1) = (1-a_1^{\,s})$ then this Definition (4) will be reduced to (m,s)-HA convex function.

Example 1: A function $f : \mathbb{I} \subset \mathbb{R}_+ \to \mathbb{R}$ is called (m,h_1,h_2,s)-HA convex function defined as $f(x) = \frac{1}{x^\alpha}$, for some fixed $\alpha \geq 1$. Let $h_1^s(a_1) = a_1^{\alpha_1}$, $h_2^s(a_1) = a_1^{\alpha_2}$, for all $a_1 \in [0,1]$ and $0 < \alpha_1, \alpha_2 \leq 1$.

Let $x, y \in \mathbb{I}$, then by definition

$$f(x) = \frac{1}{x^\alpha} \Rightarrow f\left(\frac{xy}{a_1x + m(1-a_1)y}\right) = \left(\frac{1}{\frac{xy}{a_1x + m(1-a_1)y}}\right)^\alpha,$$

$$= \left(\frac{a_1x + m(1-a_1)y}{xy}\right)^\alpha,$$

$$= \frac{[a_1x + m(1-a_1)y]^\alpha}{[xy]^\alpha},$$

$$\leq \frac{a_1x^\alpha + m^\alpha(1-a_1)y^\alpha}{[x]^\alpha[y]^\alpha}.$$

As we know that $h_1^s(a_1) = a_1^{\alpha_1}$, $h_2^s(a_1) = a_1^{\alpha_2}$ and $0 < \alpha_1, \alpha_2 \leq 1$ then $a_1 \leq a_1^{\alpha_1}$ and $(1-a_1) \leq (1-a_1^{\alpha_1})$.

Now above inequality can be written as

$$f\left(\frac{xy}{a_1x + m(1-a_1)y}\right) \leq \frac{a_1^{\alpha_1}x^\alpha + m^\alpha(1-a_1^{\alpha_1})y^\alpha}{[x]^\alpha[y]^\alpha}$$

$$\leq a_1^{\alpha_1}\frac{1}{y^\alpha} + m^\alpha(1-a_1^{\alpha_1})\frac{1}{x^\alpha}$$

$$= h_1^s(a_1)f(y) + h_2^s(a_1)f(x)$$

This showed that f is a (m,h_1,h_2,s)-HA convex function.

Theorem 5: *Let $h_1, h_2 : \mathbb{I} \to \mathbb{R}$ such that $h_1, h_2 \neq 0$. Let $f : \mathbb{I} \subset \mathbb{R}_+ \to \mathbb{R}_+$ be (m,h_1,h_2,s)-harmonically convex function. If $m = 1$ then $h_1^s(a_1) + h_2^s(a_1) \geq 1$, for all $a_1 \in [0,1]$.*

Proof: As f is a (m,h_1,h_2,s)-harmonically convex function with $m = 1$ then

$$f(x)= f\left(\frac{x^2}{a_1x+(1-a_1)x}\right),$$
$$\leq h_1^s f(x)+ h_2^s(1-a_1)f(x),$$
$$\leq [h_1^s(a_1)+ h_2^s(1-a_1)]f(x).$$

This proves that $h_1^s(a_1)+h_2^s(1-a_1)\geq 1$ for all $a_1\in[0,1]$.

Theorem 6: *Let* $h_i:[0,1]\to\mathbb{R}$ *such that* $h_i\neq 0$ *for all* $i=1,2,3,4$. *Let* $f:\mathbb{I}\subset\mathbb{R}_+\to\mathbb{R}_+$ *be* (m,h_1,h_2,s) *-harmonically convex function. If* $m\in[0,1]$, $a_1\in[0,1]$ *and* $h_1^s(a_1)\leq h_3^s(a_1)$, $h_2^s(a_1)\leq h_4^s(a_1)$ *then f is a* (m,h_3,h_4,s) *-harmonically convex function.*

Proof: As *f* is a (m,h_1,h_2,s) -harmonically convex function then

$$f\left(\frac{xy}{a_1x+m(1-a_1)y}\right)\leq h_1^s(a_1)f(x)+mh_2^s(1-a_1)f(y),$$
$$f\left(\frac{xy}{a_1x+m(1-a_1)y}\right)\leq h_3^s(a_1)f(x)+mh_4^s(1-a_1)f(y).$$

This completes the proof.

Theorem 7: *Let* $h_i:[0,1]\to\mathbb{R}$ *such that* $h_i\neq 0$ *for all* $i=1,2$. *Let* $f:\mathbb{I}\subset\mathbb{R}_+\to\mathbb{R}_+$ *and* $g:J\subset\mathbb{R}_+\to\mathbb{R}_+$ *be* (m,h_1,h_2,s) *-harmonically convex function. If f is nondecreasing function with respect to g and g is m-HA-convex function on J with* $m\in[0,1]$, $a_1\in[0,1]$ *then* $f\circ g$ *is a* (m,h_1,h_2,s) *-harmonically convex function.*

Proof: As *g* is a *m*-HA convex function and $m\in[0,1]$, $a_1\in[0,1]$ with , we have

$$g\left(\frac{xy}{a_1x+m(1-a_1)y}\right)\leq a_1g(x)+m(1-a_1)g(y).$$

As *f* is a nondecreasing (m,h_1,h_2,s) - convex function, we have

$$f\left(g\left(\frac{xy}{a_1x+m(1-a_1)y}\right)\right)\leq a_1f(g(x))+m(1-a_1)f(g(y)).$$

By using the definition of (m,h_1,h_2,s) -convex function, we get

$$g\left(\frac{xy}{a_1x+m(1-a_1)y}\right)\leq h_1^s(a_1)f(g(x))+mh_2^s(1-a_1)f(g(y)).$$

This completes the proof that $f\circ g$ is a (m,h_1,h_2,s) -harmonically convex function.

3. Hermite-Hadamard (H-H) inequality.

In this section, some extensions of H-H inequality for (m,h_1,h_2,s)-HA convex function is obtained and presented in the form of theorems.

Theorem 7: *Let* $h_1,h_2:[0,1]\to\mathbb{R}$ *such that* $h_1,h_2\neq 0.$ *Let* $f:\mathbb{I}\subset\mathbb{R}_+\to\mathbb{R}_+$ *be an* (m,h_1,h_2,s)*-harmonically convex function with* $s\in[0,1],$ $m\in[0,1],$ $a_1\in[0,1]$ *then*

$$f\left(\frac{2pq}{p+q}\right)\leq \frac{pqh_1^s\left(\frac{1}{2}\right)}{q-p}\int_p^q \frac{f(x)}{x^2}dx+\frac{pqmh_2^s\left(\frac{1}{2}\right)}{q-p}\int_p^q \frac{f(xm)}{x^2}dx. \tag{11}$$

Proof:

As we know that $\left(\frac{2pq}{p+q}\right)$ can be written as $\left(\frac{1}{\left(\frac{1}{2}\right)\left(\frac{1}{\frac{pq}{a_1p+(1-a_1)q}}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{\frac{pq}{a_1q+(1-a_1)p}}\right)}\right)$

for all $a_1\in[0,1]$ then we get

$$f\left(\frac{2pq}{p+q}\right)\leq h_1^s\left(\frac{1}{2}\right)f\left(\frac{pq}{a_1p+(1-a_1)q}\right)+mh_2^s\left(\frac{1}{2}\right)f\left(\frac{pq}{a_1p+(1-a_1)q}\right). \tag{12}$$

Put $x=\frac{pq}{a_1p+(1-a_1)q}=\frac{pq}{a_1p+(1-a_1)q}$ in Inequality (12) and integrate with respect to $a_1\in[0,1],$ we get

$$\int_0^1 f\left(\frac{pq}{a_1p+(1-a_1)q}\right)da_1=\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(x)}{x^2}dx. \tag{13}$$

$$\int_0^1 f\left(\frac{pqm}{a_1q+(1-a_1)p}\right)da_1=\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(xm)}{x^2}dx. \tag{14}$$

By substituting the values in Equation (13) and Equation (14) in Inequality (12), we get

$$f\left(\frac{2pq}{p+q}\right)\leq h_1^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(x)}{x^2}dx+mh_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(xm)}{x^2}dx. \tag{15}$$

This Inequality (15) completes the proof.

Theorem 8: *Let* $h_1,h_2:[0,1]\to\mathbb{R}$ *such that* $h_1,h_2\neq 0.$ *Let* $f:\mathbb{I}\subset\mathbb{R}_+\to\mathbb{R}_+$ *be an* (m,h_1,h_2,s)*-harmonically convex function with* $s\in[0,1],$ $m\in[0,1],$ $a_1\in[0,1]$ *then*

$$\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(x)}{x^2}dx \le min\{f(p)\int_0^1 h_1^s(a_1)da_1 + mf(qm)\int_0^1 h_2^s(1-a_1)da_1 f(q)\int_0^1 h_1^s(a_1)da_1$$
$$+ mf(pm)\int_0^1 h_2^s(1-a_1)da_1\}. \tag{16}$$

Proof: Let $p, q \in \mathbb{R}_+$ and *f* be a (m, h_1, h_2, s)-harmonically convex function, then

$$f\left(\frac{pq}{a_1p+(1-a_1)q}\right) \le h_1^s(a_1)f(p) + mh_2^s(1-a_1)f(qm). \tag{17}$$

Integrate Inequality (17) with respect to $a_1 \in [0,1]$, we get

$$\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(x)}{x^2}dx \le \int_0^1 h_1^s(a_1)da_1 f(p) + m\int_0^1 h_2^s(1-a_1)da_1 f(qm). \tag{18}$$

Now by using Inequality (17) and Inequality (18), we get

$$\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(x)}{x^2}dx \le min\{f(p)\int_0^1 h_1^s(a_1)da_1 + mf(qm)\int_0^1 h_2^s(1-a_1)da_1, f(q)\int_0^1 h_1^s(a_1)da_1$$
$$+ mf(pm)\int_0^1 h_2^s(1-a_1)da_1\}. \tag{19}$$

This Inequality (19) completes the proof.

Theorem 8: *Let* $h_1, h_2 : [0,1] \to \mathbb{R}$ *such that* $h_1, h_2 \ne 0$. *Let* $f : \mathbb{I} \subset \mathbb{R}_+ \to \mathbb{R}_+$ *be an* (m, h_1, h_2, s)*-harmonically convex function with* $s \in [0,1]$, $m \in [0,1]$, $a_1 \in [0,1]$ *then*

$$f\left(\frac{pq}{p+q}\right) \le h_1^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(x)}{x^2}dx + mh_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q \frac{f(xm)}{x^2}dx$$
$$\le min\left\{\left[h_1^s\left(\frac{1}{2}\right)f(q) + mh_2^s\left(\frac{1}{2}\right)f(mp)\right]\int_0^1 h_1^s(a_1)da_1 + m\left[h_1^s\left(\frac{1}{2}\right)f(mp)\right.\right.$$
$$\left. + mh_2^s\left(\frac{1}{2}\right)f(m^2q)\right]\int_0^1 h_2^s(a_1)da_1, \left[h_1^s\left(\frac{1}{2}\right)f(p) + mh_2^s\left(\frac{1}{2}\right)f(mq)\right]\int_0^1 h_1^s(a_1)da_1$$
$$\left.+ m\left[h_1^s\left(\frac{1}{2}\right)f(mq) + mh_2^s\left(\frac{1}{2}\right)f(m^2p)\right]\int_0^1 h_2^s(a_1)da_1\right\}. \tag{20}$$

Proof: Let $p, q \in \mathbb{R}_+$ and *f* be a (m, h_1, h_2, s)-harmonically convex function, then

$$f\left(\frac{2pq}{p+q}\right)\le h_1^s\left(\frac{1}{2}\right)f\left(\frac{pq}{a_1p+(1-a_1)q}\right)+h_2^s\left(\frac{1}{2}\right)f\left(\frac{pqm}{a_1p+(1-a_1)q}\right), \tag{21}$$

Similarly,

$$f\left(\frac{2pq}{p+q}\right)\le h_1^s\left(\frac{1}{2}\right)f\left(\frac{pq}{a_1p+(1-a_1)q}\right)+h_2^s\left(\frac{1}{2}\right)f\left(\frac{pqm}{a_1p+(1-a_1)q}\right), \tag{23}$$

$$\le h_1^s\left(\frac{1}{2}\right)[h_1^s(a_1)f(p)+mh_2^s(1-a_1)f(qm)]+mh_2^s\left(\frac{1}{2}\right)[h_1^s(a_1)f(mq)$$
$$+mh_2^s(1-a_1)f(pm^2)]. \tag{24}$$

After integrating Inequality (22) and Inequality (24) with respect to $a_1\in[0,1]$, we get

$$f\left(\frac{2pq}{p+q}\right)\le h_1^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(x)}{x^2}dx+mh_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(xm)}{x^2}dx, \tag{25}$$

$$\le h_1^s\left(\frac{1}{2}\right)\left[f(q)+mh_2^s\left(\frac{1}{2}\right)f(pm)\right]\int_0^1 h_1^s(a_1)da_1$$
$$+\left[f(pm)+mh_2^s\left(\frac{1}{2}\right)f(qm^2)\right]\int_0^1 h_2^s(a_1)da_1, \tag{26}$$

$$f\left(\frac{2pq}{p+q}\right)\le h_1^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(x)}{x^2}dx+mh_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(xm)}{x^2}dx, \tag{27}$$

$$\le h_1^s\left(\frac{1}{2}\right)\left[f(p)+mh_2^s\left(\frac{1}{2}\right)f(qm)\right]\int_0^1 h_1^s(a_1)da_1$$
$$+\left[f(qm)+mh_2^s\left(\frac{1}{2}\right)f(pm^2)\right]\int_0^1 h_2^s(a_1)da_1, \tag{28}$$

respectively.

Now by using Inequality (26) and Inequality (28), we get

$$f\left(\frac{pq}{p+q}\right)\le h_1^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(x)}{x^2}dx+mh_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(xm)}{x^2}dx$$
$$\le min\left\{\left[h_1^s\left(\frac{1}{2}\right)f(q)+mh_2^s\left(\frac{1}{2}\right)f(mp)\right]\int_0^1 h_1^s(a_1)da_1+m\left[h_1^s\left(\frac{1}{2}\right)f(mp)\right.\right.$$
$$\left.+mh_2^s\left(\frac{1}{2}\right)f(m^2q)\right]\int_0^1 h_2^s(a_1)da_1,\left[h_1^s\left(\frac{1}{2}\right)f(p)+mh_2^s\left(\frac{1}{2}\right)f(mq)\right]\int_0^1 h_1^s(a_1)da_1$$
$$\left.+m\left[h_1^s\left(\frac{1}{2}\right)f(mq)+mh_2^s\left(\frac{1}{2}\right)f(m^2p)\right]\int_0^1 h_2^s(a_1)da_1\right\}. \tag{29}$$

Inequality (29) completes the proof.

Remarks:

(1) If we put $h_1^s(a_1)=h_2^s(a_1)=h(a_1)$ in Inequality (29), we get Inequality (7).

(2) If we put $h_1^s(a_1)=h_1(a_1)$ and $h_2^s(a_1)=h_2(a_1)$ in Inequality (29), we get the results for (m,h_1,h_2)-HA convex function.

(3) If we put $h_1^s(a_1)=a_1$ and $h_2^s(a_1)=(1-a_1)$ in Inequality (29), we get the results for m-HA convex function.

Theorem 9: *Let* $h_1,h_2:[0,1]\to\mathbb{R}$ *such that* $h_1,h_2\neq 0$. *Let* $f,g:\mathbb{I}\subset\mathbb{R}_+\to\mathbb{R}_+$ *be* (m,h_1,h_2,s)*-harmonically convex function with* $s\in[0,1]$, $m\in[0,1]$, $a_1\in[0,1]$ *then*

$$\frac{p-q}{pq}f\left(\frac{2pq}{p+q}\right)g\left(\frac{2pq}{p+q}\right)\leq\left[h_1^s\left(\frac{1}{2}\right)\right]^2\int_p^q\frac{f(x)g(x)}{x^2}dx+mh_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)\int_p^q\frac{g(xm)f(x)}{x^2}dx$$
$$+mh_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(xm)g(x)}{x^2}dx+\left[h_2^s\left(\frac{1}{2}\right)\right]^2\int_p^q\frac{g(xm)f(xm)}{x^2}dx. \quad (20)$$

Proof: Let $p,q\in\mathbb{R}_+$ and be a (m,h_1,h_2,s)-harmonically convex function, then

$$f\left(\frac{2pq}{p+q}\right)g\left(\frac{2pq}{p+q}\right)\leq\left[h_1^s\left(\frac{1}{2}\right)f\left(\frac{pqm}{a_1p+(1-a_1)q}\right)+mh_2^s\left(\frac{1}{2}\right)f\left(\frac{pqm}{a_1p+(1-a_1)q}\right)\right]$$
$$\left[h_1^s\left(\frac{1}{2}\right)g\left(\frac{pq}{a_1p+(1-a_1)q}\right)+mh_2^s\left(\frac{1}{2}\right)g\left(\frac{pq}{a_1p+(1-a_1)q}\right)\right]. \quad (21)$$

$$=\left[\left[h_1^s\left(\frac{1}{2}\right)\right]^2f\left(\frac{pqm}{a_1p+(1-a_1)q}\right)g\left(\frac{pq}{a_1p+(1-a_1)q}\right)\right.$$
$$+mh_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)f\left(\frac{pq}{a_1p+(1-a_1)q}\right)g\left(\frac{pqm}{a_1p+(1-a_1)q}\right)$$
$$+mh_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)f\left(\frac{pqm}{a_1p+(1-a_1)q}\right)g\left(\frac{pq}{a_1p+(1-a_1)q}\right)$$
$$\left.+\left[mh_2^s\left(\frac{1}{2}\right)\right]^2f\left(\frac{pqm}{a_1p+(1-a_1)q}\right)g\left(\frac{pqm}{a_1p+(1-a_1)q}\right)\right]. \quad (22)$$

Integrate Inequality (22) with respect to $a_1\in[0,1]$, we get

$$\frac{p-q}{pq}f\left(\frac{2pq}{p+q}\right)g\left(\frac{2pq}{p+q}\right)\leq\left[h_1^s\left(\frac{1}{2}\right)\right]^2\int_p^q\frac{f(x)g(x)}{x^2}dx+mh_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)\int_p^q\frac{g(xm)f(x)}{x^2}dx$$
$$+mh_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(xm)g(x)}{x^2}dx+\left[h_2^s\left(\frac{1}{2}\right)\right]^2\int_p^q\frac{g(xm)f(xm)}{x^2}dx.$$
(23)

Inequality (23) completes the required proof.

4. Application for Mean (Average).

As we know that for all $p,q \in \mathbb{R}$, some means (average) are as follows

(1) The geometric mean

$$G(p,q)=\sqrt{pq}.$$

(2) The harmonic mean

$$H(p,q)=\frac{2pq}{p+q}.$$

(3) The α-logarithmic mean

$$L_n(p,q)=\left[\frac{q^{\alpha+1}-p^{\alpha+1}}{(\alpha+1)(q-p)}\right]^{\frac{1}{\alpha}}.$$

Here, $\alpha \in \mathbb{R}\setminus\{0\}$ and $p<q$.

In next Theorem (10), we present some relationship between the previous results and the means (averages) defined above.

Theorem 10: *Let* $0<p<q$. *Then*

$$2^{(\alpha_1+\alpha_2)}G^2(p^\alpha,q^\alpha)H^{-\alpha}(p,q)\le [2^{\alpha_2}m^{\alpha-1}+2^{\alpha_1}]L_\alpha^\alpha(p,q). \tag{24}$$

Proof: Let $p,q\in\mathbb{R}_+$ and f be a (m,h_1,h_2,s)-harmonically convex function, then by using Example (1) as $f(x)=\frac{1}{x^\alpha}$ for all $\alpha\ge 1$ and $h_1^s(a_1)=a_1^{\alpha_1}$, $h_2^s(a_1)=a_1^{\alpha_2}$, for all $a_1\in[0,1]$ and $0<\alpha_1,\ \alpha_2\le 1$. Further, by using Theorem (7), we get

$$f\left(\frac{2pq}{p+q}\right)\le \frac{pqh_1^s\left(\frac{1}{2}\right)}{q-p}\int_p^q \frac{f(x)}{x^2}dx+\frac{pqmh_2^s\left(\frac{1}{2}\right)}{q-p}\int_p^q \frac{f(xm)}{x^2}dx. \tag{25}$$

After solving Inequality (25) using the mean (averages), we get

$$f\left(\frac{2pq}{p+q}\right)=\left(\frac{2pq}{p+q}\right)^{-\alpha}=H^{-\alpha}(p,q), \tag{26}$$

$$h_1^s\left(\frac{1}{2}\right)=\frac{1}{2^{\alpha_1}}=2^{-\alpha_1}, \tag{27}$$

Similarly, $$h_2^s\left(\frac{1}{2}\right)=\frac{1}{2^{\alpha_2}}=2^{-\alpha_2}, \tag{28}$$

$$\begin{aligned}\left(\frac{pq}{q-p}\right)\int_p^q\frac{f(x)}{x^2}dx &=\left(\frac{pq}{q-p}\right)\int_p^q\frac{1}{x^{\alpha+2}}dx, \quad (29)\\ &=\left(\frac{pq}{-(q-p)(\alpha+1)}\right)\left[q^{-(\alpha+1)}-p^{-(\alpha+1)}\right],\\ &=\left(\frac{pq}{(q-p)(\alpha+1)}\right)\left[\frac{1}{p^{(\alpha+1)}}-\frac{1}{q^{(\alpha+1)}}\right],\\ &=\left(\frac{pq}{(q-p)(\alpha+1)}\right)\left(\frac{q^{(\alpha+1)}-p^{(\alpha+1)}}{q^{(\alpha+1)}p^{(\alpha+1)}}\right),\\ &=\left(\frac{1}{q^\alpha p^\alpha}\right)\left(\frac{q^{(\alpha+1)}-p^{(\alpha+1)}}{(\alpha+1)(q-p)}\right),\\ &=\left(\frac{1}{G^2(q^\alpha,p^\alpha)}\right)\left\{\left(\frac{q^{(\alpha+1)}-p^{(\alpha+1)}}{(\alpha+1)(q-p)}\right)^{\frac{1}{\alpha}}\right\}^\alpha,\\ &=\left(\frac{1}{G^2(q^\alpha,p^\alpha)}\right)L_\alpha^\alpha(p,q). \quad (30)\end{aligned}$$

Similarly,

$$\begin{aligned}\left(\frac{pqm}{q-p}\right)\int_p^q\frac{f(xm)}{x^2}dx &=\left(\frac{pq}{q-p}\right)\int_p^q\frac{1}{m^\alpha x^{\alpha+2}}dx,\\ &=\left(\frac{pq}{-(q-p)(\alpha+1)m^{(\alpha-1)}}\right)\left[q^{-(\alpha+1)}-p^{-(\alpha+1)}\right],\\ &=\left(\frac{pq}{(q-p)(\alpha+1)m^{(\alpha-1)}}\right)\left[\frac{1}{p^{(\alpha+1)}}-\frac{1}{q^{(\alpha+1)}}\right],\\ &=\left(\frac{pq}{(q-p)(\alpha+1)m^{(\alpha-1)}}\right)\left(\frac{q^{(\alpha+1)}-p^{(\alpha+1)}}{q^{(\alpha+1)}p^{(\alpha+1)}}\right),\\ &=\left(\frac{1}{q^\alpha p^\alpha m^{(\alpha-1)}}\right)\left(\frac{q^{(\alpha+1)}-p^{(\alpha+1)}}{(\alpha+1)(q-p)}\right),\\ &=\left(\frac{1}{G^2(q^\alpha,p^\alpha)m^{(\alpha-1)}}\right)\left\{\left(\frac{q^{(\alpha+1)}-p^{(\alpha+1)}}{(\alpha+1)(q-p)}\right)^{\frac{1}{\alpha}}\right\}^\alpha,\\ &=\left(\frac{1}{G^2(q^\alpha,p^\alpha)m^{(\alpha-1)}}\right)L_\alpha^\alpha(p,q). \quad (31)\end{aligned}$$

After substituting the values from Equation (30) and Equation (31) in Inequality (25), we get

$$H^{-\alpha}(p,q)\leq\left(\frac{1}{2^{\alpha_1}}\right)\left(\frac{1}{G^2(q^\alpha,p^\alpha)}\right)L_\alpha^\alpha(p,q)+\left(\frac{1}{2^{\alpha_2}}\right)\left(\frac{1}{G^2(q^\alpha,p^\alpha)m^{(\alpha-1)}}\right)L_\alpha^\alpha(p,q). \quad (32)$$

$$=\frac{2^{\alpha_2}m^{(\alpha-1)}+2^{\alpha_1}}{2^{\alpha_1+\alpha_2}m^{(\alpha-1)}}\left(\frac{1}{G^2(q^\alpha,p^\alpha)}\right)L_\alpha^\alpha(p,q). \quad (33)$$

Inequality (33) can be written as

$$2^{(\alpha_1+\alpha_2)}G^2(p^\alpha,q^\alpha)H^{-\alpha}(p,q)\leq[2^{\alpha_2}m^{\alpha-1}+2^{\alpha_1}]L_\alpha^\alpha(p,q). \quad (34)$$

Inequality (34) completes the proof.

5. Conclusion

This paper utilized the procedure of combining more than two functions to extend the convex function. Different types of convex functions are used to extend the previous results and to investigate the H-H inequalities. For some specific value of h, h^s, m and α, almost all the previous results for discussed functions are derived. The comparison between new as well as old results reflects that all the previous results can be obtained for these new results by only choosing some specific conditions. H-H and Fejer's inequalities are also used to produce the results and original H-H inequalities can also be found by using these results. On the basis of discussed inequalities, mathematical means (averages) are also applied in calculating the required results.

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