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A FLEXIBLE INVERSE LOMAX WEIBULL-WEIBULL MODEL FOR SURVIVAL AND RELIABILITY DATA

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ABSTRACT

In this paper, an extension of the Weibull distribution using Inverse Lomax Weibull-G (ILWG) called Inverse Lomax Weibull-Weibull Distribution (ILWWD) was introduced. The statistical properties of the ILWWD were also derived. This includes the moment, moment generating function, quantile function, and distributions of order statistics. The parameters of the ILWWD were estimated using the maximum likelihood estimates (MLE) method. Simulation studies were presented for three distinct cases. Survival regression was also presented. Goodness-of-fit statistics were employed to assess the effectiveness of the ILWWD and Log-ILWWD. This indicated that the proposed distribution is superior in fitting leukemia data. Lastly, the flexibility of the ILWWD was demonstrated using a real-life dataset. These suggest that the proposed distribution is superior in fitting the datasets presented in this study.

Keywords: cumulative distribution function, Inverse Lomax-G, Inverse Lomax distribution, probability density function, Weibull distribution.

INTRODUCTION

As foundational ideas in statistics, statistical distributions offer a framework for comprehending the properties and behavior of data. Within a dataset, these distributions characterize the likelihood of witnessing certain values or outcomes. Notably, the Weibull distribution is widely employed in many domains, especially survival analysis and reliability engineering. It is a powerful tool for predicting failure rates and analyzing lifetime characteristics as it offers a flexible model for expressing the period until mechanical or electronic component breakdown (Lai et al., 2006). Furthermore, the Weibull distribution, being an extreme value distribution, is particularly good at predicting extreme occurrences. This includes the highest amount of rainfall in a single day and other extreme events, which helps with risk assessment and hazard prediction (Menčík, 2016). Its versatility further highlights its significance in various scientific fields by enabling a broad range of applications outside dependability, such as environmental factor modeling and wind energy analysis (Kızılersü et al., 2018).

However, there are drawbacks to the Weibull distribution as well. The presumption of a monotonically increasing or decreasing hazard function is one of the drawbacks of the Weibull distribution; real-world data may not always support this assumption. If the hazard rate displays non-monotonic behavior or complicated patterns, other distributions might be a better fit. Furthermore, it is occasionally challenging to estimate the parameters of the Weibull distribution accurately, especially when there are small samples or when the underlying assumptions are broken. The Weibull distribution is nevertheless a popular and useful tool in statistical modeling, providing insights into the survival and dependability traits of various systems and events, despite these disadvantages.

Scholars consistently put forth novel extensions of the Weibull distribution. The purpose of these extensions is to increase goodness-of-fit and overcome conventional model limitations. Silva et al. (2010) proposed a Log-Weibull extended distribution that can be employed to analyze data with bathtub-shaped failure rates. This model is based on an extended Weibull distribution. The authors investigated parameter estimation techniques such as maximum likelihood and Bayesian approaches, and this model can oversee censored data. Additionally, they examined influence diagnostics to determine the impact of specific data points on the model's projections. The applicability of the model is demonstrated in the paper's conclusion through simulations and a real-data example. At the same time, a new probability distribution called the log-Beta Weibull distribution has been proposed by Ortega et al. (2013) to analyze event's lifetimes or durations. The log-Beta Weibull distribution's flexibility allows it to reflect various failure rate shapes, which include increasing, bathtub, decreasing, and constant. Based on the log-Beta distribution, the authors suggest a regression model to forecast the recurrence of prostate cancer and validate its applicability using an actual data set.

Cruz et al. (2016) proposed a novel regression model for survival analysis that is based on the odd log-logistic Weibull distribution. In contrast to the conventional log-Weibull model, this model can oversee data with different failure rate forms. The authors discuss how to estimate the parameters of the model, examine how it affects estimations, and use residuals to assess the goodness-of-fit. They also use an actual data set to demonstrate how the model may be utilized. The transmuted Kumaraswamy-Weibull (TKWD) distribution is a novel distribution that was proposed by Khan et al. (2020) for reliability data analysis. The TKWD is based on a mathematical method known as the quadratic rank transmutation map (QRTM). Some of the mathematical properties of this new distribution were examined. Additionally, this study introduces a regression model for analyzing lifetime data based on the TKWD distribution. With consideration for factors that could affect those failures, this model can be employed to estimate the chance of failures over time. The authors go on how to use a statistical technique known as maximum likelihood estimation (MLE) to estimate the parameters of this model. Lastly, two real-world examples are provided to illustrate the applicability of the regression model and the TKWD distribution. In addition, the Extended Exponential Weibull (ExEW) distribution is a novel probability distribution for survival analysis that was presented by Braima (2024). Compared to the conventional Weibull distribution, the ExEW distribution provides greater flexibility by supporting multiple hazard rate forms. Additionally, the authors presented an extended model called ExEW-AFT that uses an accelerated failure time framework to analyze the variables. This makes it possible to include variables that could affect survival times. Both approaches' usefulness was illustrated using real-world datasets and simulation studies. Despite numerous extensions, current Weibull-based distributions remain limited in capturing certain tail behaviors, thus justifying the need for a model with enhanced flexibility, like Inverse Lomax Weibull-Weibull Distribution (ILWWD).

Existing Weibull-based distributions, while effective for many reliability applications, struggle to model data with non-monotonic hazard functions, leading to suboptimal fits in scenarios like the leukemia data. This study addresses the need for a more flexible distribution that can better model diverse hazard shapes in complex datasets, thereby improving model accuracy and predictive power in fields such as biomedical research and engineering. The objectives of this paper are:

i. To extend the Weibull distribution by deriving the ILWWD, providing a theoretical basis for its improved flexibility and fit.
ii. To derive and analyze the statistical properties of ILWWD, including its probability density function (PDF), cumulative distribution function (CDF), and hazard functions, and evaluate its versatility in modeling different data characteristics.
iii. To evaluate the performance of ILWWD's MLE through simulation studies, analyzing bias and Root Mean Squared Error (RMSE) across sample sizes.
iv. To apply the ILWWD model to real-life datasets, assessing its goodness-of-fit and predictive accuracy in comparison with alternative distributions.
v. To demonstrate the efficacy of ILWWD in survival regression modeling, specifically for datasets with non-monotonic hazard rates.

METHODOLOGY

The Inverse Lomax Weibull-Weibull Distribution (ILWWD)

Falgore and Doguwa (2020) proposed an Inverse Lomax-G family of distribution with CDF given by:

$$F\left(x;\gamma,\lambda,\varpi\right)=\left[1+\lambda\left(\frac{\left(1-G\left(x;\varpi\right)\right)}{G\left(x;\varpi\right)}\right)\right]^{-\gamma};x,\gamma,\lambda,\varpi>0. \tag{1}$$

If $G(x;\varpi)$ is the CDF of the Weibull-G family of distributions of Bourguignon et al. (2014), then by setting the scale parameter to unity, the CDF of the Weibull-G can be given by

$$G(x;\alpha,\varpi)=1-e^{-\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]^{\alpha}};x,\alpha>0. \tag{2}$$

Then, based on Equation (1) and Equation (2), the CDF of the Inverse Lomax Weibull-G (ILWG) can be given as

$$F(x;\alpha,\gamma,\lambda,\zeta)=\left[1+\lambda\left(\frac{e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}\right)\right]^{-\gamma};x,\alpha,\gamma,\lambda,\zeta>0. \tag{3}$$

Here, α and γ are the shape parameters, λ is the scale parameter, and ζ is the vector of the parameter(s) of the baseline distribution. The PDF corresponding to Equation (3) is

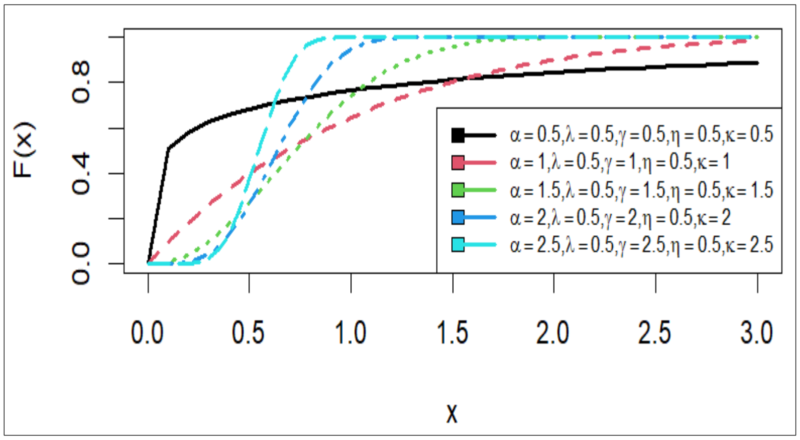
$$f(x;\alpha,\gamma,\lambda,\zeta)=\frac{\alpha\gamma\lambda g(x;\zeta)\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha-1}e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}\left[1+\lambda\left(\frac{e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}\right)\right]^{-\gamma-1}}{\left[1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}\right]^{2}\left[1-G(x;\zeta)\right]^{2}};x,\alpha,\gamma,\lambda,\zeta>0. \tag{4}$$

Using Weibull distribution of this form $G(x;\eta,\kappa)=1-e^{-(\eta x)^{\kappa}}$ and $g(x;\eta,\kappa)=\kappa\eta^{\kappa}x^{\kappa-1}e^{-(\eta x)^{\kappa}}$ as the baseline, we have the CDF and PDF of the ILWWD as

$$F(x;\eta,\kappa,\alpha,\gamma,\lambda)=\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]^{-\gamma};x,\eta,\kappa,\alpha,\gamma,\lambda>0. \tag{5}$$

Figure 1

The CDF of the ILWWD at Various Values of the Parameters

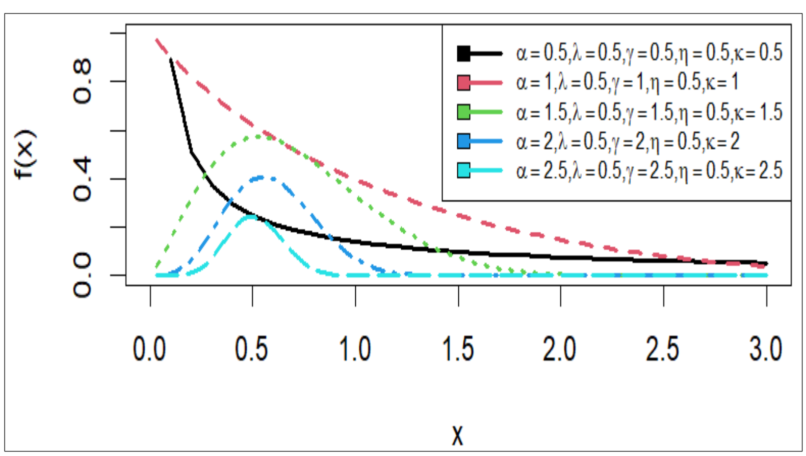


$$f(x;\eta,\kappa,\alpha,\gamma,\lambda)=\frac{\alpha\gamma\lambda\kappa\eta^{\kappa}x^{\kappa-1}e^{(\eta x)^{\kappa}}\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha-1}e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]^{-\gamma-1}}{\left[1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right]^{2}};x,\eta,\kappa,\alpha,\gamma,\lambda>0 \tag{6}$$

where η and λ are the scale parameters, while κ, γ, α are the shape parameters.

Figure 2

The PDF of ILWWD at Different Various Parameter Values



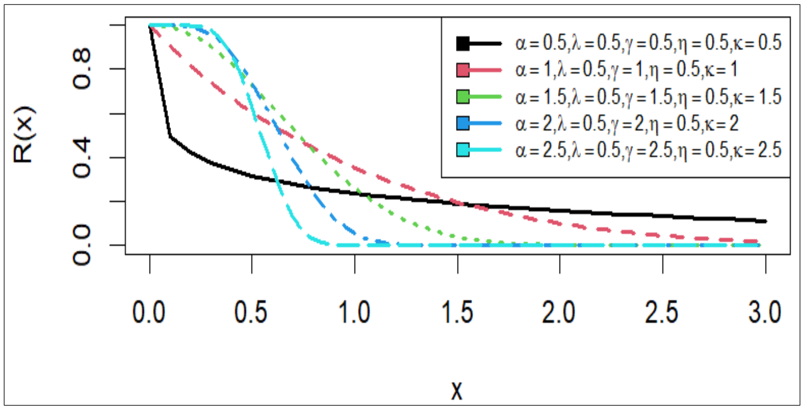
The Statistical Properties of the ILWWD

The survival function and hazard function of the ILWWD are given by:

$$S(x;\eta,\kappa,\alpha,\gamma,\lambda)=1-\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]^{-\gamma};x,\eta,\kappa,\alpha,\gamma,\lambda>0. \tag{7}$$

Figure 3

The Survival Function Plot of the ILWWD



$$h(x;\eta,\kappa,\alpha,\gamma,\lambda)=\frac{\alpha\gamma\lambda\kappa\eta^{\kappa}x^{\kappa-1}e^{(\eta x)^{\kappa}}\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha-1}e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]^{-\gamma-1}}{1-\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]^{-\gamma}\left[1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right]^{2}};x,\eta,\kappa,\alpha,\gamma,\lambda>0 \tag{8}$$

Figure 4

The Hazard Function of the ILWWD with Different Shapes

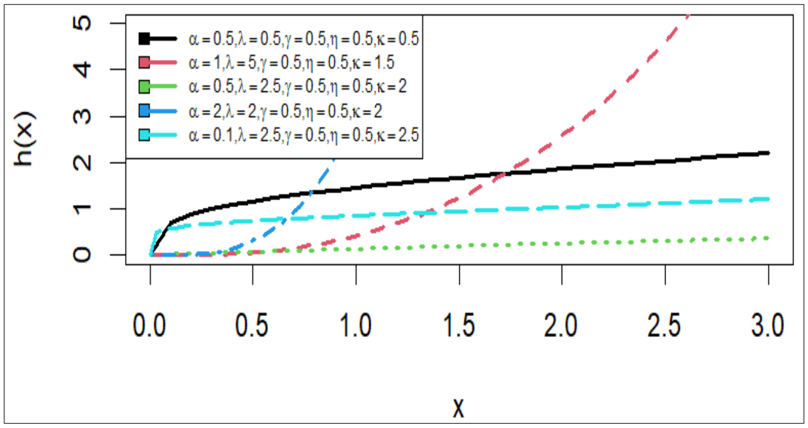


Figure 1 displays the CDF of the ILWWD at various values of the parameters, which is an indication of a valid CDF as it starts from 0 and ends at 1. Figure 2 indicates the different shapes that ILWWD has, which include symmetry and skew to the right. Figure 3 presents the survival function of the ILWWD, which indicates that the survival probability starts at 1 and decays towards zero over time. Moreover, Figure 4 presents different shapes of the hazard function of the ILWWD, which include constant, increasing, and decreasing.

The Statistical Properties of the ILWWD

The quantile function of the ILWWD is given by:

$$x=\frac{\left(\log\left\{1+\left[\log\left(\frac{u^{\frac{-1}{\gamma}}+\lambda-1}{u^{\frac{-1}{\gamma}}-1}\right)\right]^{\frac{1}{\alpha}}\right\}\right)^{\frac{1}{\kappa}}}{\eta}. \tag{9}$$

The r^{th} moment of the origin of the ILWWD is given by

$$E\left(X^{r}\right)=\Omega_{B}\frac{(A\alpha)^{-(r+\kappa)}\Gamma\left(\frac{r+\kappa}{\kappa}\right)}{\kappa}, \tag{10}$$

where

$$\Omega=\alpha\gamma\lambda\kappa\alpha^{\kappa}\sum_{i,j,l,n,p,d,q,m,c,f,h,s=0}^{\infty}\frac{(-1)^{j+l+n+p+m+q+c+f+h+s}\,f^{h}i^{j}m^{n}}{n!q!h!}\binom{\alpha j}{l}\binom{i}{m}\binom{\alpha j}{l}\binom{\alpha m}{p}\binom{\alpha q}{c}\binom{\alpha-1}{d}\binom{-2}{f}\binom{-(1+\gamma)}{i}\binom{\alpha h}{s}$$

$A=l-\alpha j+p-m\alpha+c-\alpha q+d-\alpha+2$, $B=i,j,l,n,p,d,f,m,c,t,h,s$, and $r=1,2,3,4,\ldots$

The moment-generating function (mgf) of the ILWWD is given by

$$\sum_{r=0}^{\infty}\frac{t^{r}\Omega_{B}(A\alpha)^{-(r+\kappa)}\Gamma\left(\frac{r+\kappa}{\kappa}\right)}{r!\kappa}. \tag{11}$$

The distribution of order statistics for the ILWWD can be expressed as:

$$f_{j:n}\left(x;\alpha,\gamma,\lambda,\eta,\kappa\right)=\frac{n!\alpha\gamma\lambda\kappa\alpha^{\kappa}x^{\kappa-1}e^{(\eta x)^{\kappa}}}{\left(j-1\right)!\left(n-j\right)!}\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha-1}e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}\left(1-e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}\right)^{-2}$$

$$\times\left(1+\lambda\left(\frac{e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}{1-e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}\right)\right)^{-(\gamma j+1)}\left(1-\left[1+\lambda\left(\frac{e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}{1-e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}\right)\right]^{-\gamma}\right)^{(n-j)}. \tag{12}$$

Also, the distributions of the 1^{st} and n^{th} order statistics are given by:

$$f_{1:n}\left(x;\alpha,\gamma,\lambda,\eta,\kappa\right)=\frac{n\alpha\gamma\lambda\kappa\alpha^{\kappa}x^{\kappa-1}e^{(\eta x)^{\kappa}}}{\left(1-e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}\right)^{2}}\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha-1}e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}$$

$$\times\left(1+\lambda\left(\frac{e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}{1-e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}\right)\right)^{-(\gamma+1)}\left(1-\left[1+\lambda\left(\frac{e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}{1-e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}\right)\right]^{-\gamma}\right)^{(n-1)}, \tag{13}$$

$$f_{n:n}\left(x;\alpha,\gamma,\lambda,\eta,\kappa\right)=\frac{n\alpha\gamma\lambda\kappa\alpha^{\kappa}x^{\kappa-1}e^{(\eta x)^{\kappa}}}{\left(1-e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}\right)^{2}}\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha-1}e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}\left(1+\lambda\left(\frac{e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}{1-e^{-\left(\frac{1-e^{-(\eta x)^{\kappa}}}{e^{-(\eta x)^{\kappa}}}\right)^{\alpha}}}\right)\right)^{-(\gamma n+1)}. \tag{14}$$

The Maximum Likelihood Estimation (MLE)

The maximum likelihood of the ILWWD can be computed through the maximization of log-likelihood given as

$$\log L = n\log(\alpha\gamma\lambda\kappa\eta^{\kappa}) + (\kappa-1)\sum_{i=1}^{n}\log(x) + \sum_{i=1}^{n}{}^{(\eta x)^{\kappa}} + (\alpha-1)\sum_{i=1}^{n}\log\left(\left[e^{(\eta x)^{\kappa}}-1\right]\right) - \sum_{i=1}^{n}\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}$$
$$-(\gamma+1)\sum_{i=1}^{n}\log\left(\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]\right) - 2\sum_{i=1}^{n}\log\left(\left[1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right]\right) \tag{15}$$

Alternatively, the MLEs can be derived by taking the partial derivatives of Equation (15) with respect to each parameter as follows:

$$\frac{\delta \log L}{\delta\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(\left[e^{(\eta x)^{\kappa}}-1\right]\right) - \sum_{i=1}^{n}\log\left(\left[e^{(\eta x)^{\kappa}}-1\right]\right)\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha} - \gamma\sum_{i=1}^{n}\frac{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}\log\left(\left[e^{(\eta x)^{\kappa}}-1\right]\right)e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]\left(1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right)^{2}}$$
$$-2\sum_{i=1}^{n}\frac{\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}\log\left(\left[e^{(\eta x)^{\kappa}}-1\right]\right)e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{\left(1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right)} \tag{16}$$

$$\frac{\delta \log L}{\delta\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n}\log\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right], \tag{17}$$

$$\frac{\delta \log L}{\delta\lambda} = \frac{n}{\lambda} - \gamma\sum_{i=1}^{n}\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{\left(1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right)\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]}, \tag{18}$$

$$\frac{\delta \log L}{\delta \eta} = \frac{n\kappa}{\eta} + \kappa\eta^{\kappa-1}\sum_{i=1}^{n} x^{\kappa} + (\alpha-1)\sum_{i=1}^{n}\frac{\eta^{\kappa}\kappa x^{k-1}e^{(\eta x)^{\kappa}}}{\left[e^{(\eta x)^{\kappa}}-1\right]} - \alpha\sum_{i=1}^{n}\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha-1}\eta^{\kappa}\kappa x^{k-1}e^{(\eta x)^{\kappa}}$$
$$+\gamma\lambda\sum_{i=1}^{n}\frac{\alpha\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha-1}\eta^{\kappa}\kappa x^{k-1}e^{(\eta x)^{\kappa}}}{\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]\left[1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right]^{2}} - 2\sum_{i=1}^{n}\frac{\alpha\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha-1}\eta^{\kappa}\kappa x^{k-1}e^{(\eta x)^{\kappa}}}{\left[1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right]}, \tag{19}$$

$$\frac{\delta \log L}{\delta \kappa} = \frac{n(\eta^{\kappa}+\kappa\eta^{\kappa}\log(\eta))}{\kappa\eta^{\kappa}} + \sum_{i=1}^{n}(\eta x)^{\kappa}\log(\eta x) + (\alpha-1)\sum_{i=1}^{n}\frac{(\eta x)^{\kappa}\log(\eta x)e^{(\eta x)^{\kappa}}}{\left[e^{(\eta x)^{\kappa}}-1\right]} - \sum_{i=1}^{n}\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha-1}(\eta x)^{\kappa}\log(\eta x)e^{(\eta x)^{\kappa}}$$
$$+\gamma\lambda\alpha\sum_{i=1}^{n}\frac{\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha-1}(\eta x)^{\kappa}\log(\eta x)e^{(\eta x)^{\kappa}}}{\left[1+\lambda\left(\frac{e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}}\right)\right]\left(1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}\right)^{2}} - 2\alpha\sum_{i=1}^{n}\frac{\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha-1}(\eta x)^{\kappa}\log(\eta x)e^{(\eta x)^{\kappa}}}{1-e^{-\left[e^{(\eta x)^{\kappa}}-1\right]^{\alpha}}} \tag{20}$$

The partial derivatives in Equation (16) through Equation (20) are for each parameter. The equations are non-linear in parameters. Hence, it cannot be solved directly. Newton-Raphson approach was used to estimate the parameters in R programming software.

The Survival Regression of the Inverse Lomax Weibull-Weibull Distribution (ILWWD)

The survival regression can be derived based on Equation (6). Let $y = \log(x)$, $\eta = e^{-\mu}$, and $\kappa = \frac{1}{\sigma}$. Moreover, let y follow log-ILWWD with parameters $(\alpha, \gamma, \lambda, \mu, \sigma)$. Then, the transformed PDF and survival function of the log-ILWWD can be given as:

Finally, after some simplifications, we have

$$f(y;\sigma,\mu,\alpha,\gamma,\lambda)=\frac{\alpha\gamma\lambda e^{\left(\frac{y-\mu}{\sigma}\right)}\left[e^{e^{\left(\frac{y-\mu}{\sigma}\right)}}-1\right]^{\alpha-1}e^{-\left[e^{e^{\left(\frac{y-\mu}{\sigma}\right)}}-1\right]^{\alpha}}\left[1+\lambda\left(\frac{e^{-\left[e^{e^{\left(\frac{y-\mu}{\sigma}\right)}}-1\right]^{\alpha}}}{1-e^{-\left[e^{e^{\left(\frac{y-\mu}{\sigma}\right)}}-1\right]^{\alpha}}}\right)\right]^{-\gamma-1}}{\left[1-e^{-\left[e^{e^{\left(\frac{y-\mu}{\sigma}\right)}}-1\right]^{\alpha}}\right]^{2}};y,\sigma,\mu,\alpha,\gamma,\lambda>0 \tag{21}$$

and

$$S(y;\sigma,\mu,\alpha,\gamma,\lambda)=1-\left[1+\lambda\left(\frac{e^{-\left[e^{e^{\left(\frac{y-\mu}{\sigma}\right)}}-1\right]^{\alpha}}}{1-e^{-\left[e^{e^{\left(\frac{y-\mu}{\sigma}\right)}}-1\right]^{\alpha}}}\right)\right]^{-\gamma};y,\sigma,\mu,\alpha,\gamma,\lambda>0\,. \tag{22}$$

Also, let us define the standardized random variable z as $z=e^{\left(\frac{y-\mu}{\sigma}\right)}$. Then, Equation (21) can be re-written as

$$f(z;\sigma,\mu,\alpha,\gamma,\lambda)=\frac{\alpha\gamma\lambda e^{z}\left[e^{e^{z}}-1\right]^{\alpha-1}e^{-\left[e^{e^{z}}-1\right]^{\alpha}}\left[1+\lambda\left(\frac{e^{-\left[e^{e^{z}}-1\right]^{\alpha}}}{1-e^{-\left[e^{e^{z}}-1\right]^{\alpha}}}\right)\right]^{-\gamma-1}}{\left[1-e^{-\left[e^{e^{z}}-1\right]^{\alpha}}\right]^{2}};z,\sigma,\kappa,\alpha,\gamma,\lambda>0 \tag{23}$$

The log-linear regression model that connects the dependent variable y_i and the independent variables $m_{i1},m_{i2},m_{i3},...,m_{ip}$ is given by

$$y_i=\mu_i+\sigma z_i,\ \text{i=1,2, 3, …, n,} \tag{24}$$

where z_i is the random error with density function given in Equation (23) with $\mu_i = 0$ and $\sigma = 1$, $\mu_i = \sum_{j=1}^{p} m_{ij}\tau_j$, $\tau = \left(\tau_1, \tau_2, \tau_3, ..., \tau_p\right)^1$ is the vector associated with the independent variables. Equation (23) can be used to fit different kinds of data in which the independent variables have significant effects on the average of the dependent variable.

RESULTS

Simulation Studies of the ILWWD

To assess the effectiveness of the MLEs for the ILWWD, the following simulation studies were performed:

i. Using Equation (9), produce *s* samples of size *n*

ii. Compute the MLEs for the *s* samples, say $\left(\hat{\alpha}, \hat{\gamma}\right)$ for $i = 1, 2, 3, ..., s$

iii. Compute the biases and root mean squared errors RMSEs by:

$$Bias_{\Psi}(n) = \frac{\sum_{i=1}^{s}\left(\hat{\Psi}_i - \Psi\right)}{s}, \tag{25}$$

and

$$RMSE_{\Psi}(n) = \sqrt{\frac{\sum_{i=1}^{s}\left(\hat{\Psi}_i - \Psi\right)^2}{s}}. \tag{26}$$

$\Psi = \alpha, \gamma, \lambda.\eta, \kappa.$ The number of replications (s) considered is 10,000, and the R package was used for the simulation. The seed for the sampling was set at 123 for repetition. Three cases were considered for the initial guesses. Case I: (5,2.5,0.5,0.5,1.6), Case II: (5, 0.5, 1.5, 0.5, 1.6), and Case III: (1.6, 2.5, 0.5, 0.5, 4). The sample sizes considered for the three cases remain 25, 50, 100, 200, and 500. This represents the small, medium and large sample sizes. The simulation results are presented in Table 1 to Table 3.

Table 1

Simulation Results for Case I (5, 2.5, 0.5, 0.5, 1.6)

Sample Size	Estimates	Bias	RMSE	Sample Size	Estimates	Bias	RMSE
25	5.33	0.33	0.35	200	5.21	0.21	0.21
	2.67	0.17	0.18		2.66	0.16	0.16
	0.48	-0.02	0.15		0.63	0.13	0.14
	0.54	0.04	0.04		0.54	0.04	0.04
	1.67	0.07	0.07		1.67	0.07	0.07
50	5.27	0.27	0.29	500	5.21	0.21	0.21
	2.67	0.17	0.18		2.66	0.16	0.16
	0.55	0.05	0.15		0.63	0.13	0.13
	0.54	0.04	0.04		0.54	0.04	0.04
	1.67	0.07	0.07		1.67	0.07	0.07
100	5.27	0.22	0.23				
	2.67	0.17	0.17				
	0.62	0.12	0.14				
	0.54	0.04	0.04				
	1.67	0.07	0.07				

Table 2

Simulation Results for Case II (5, 0.5, 1.5, 0.5, 1.6)

Sample Size	Estimates	Bias	RMSE	Sample Size	Estimates	Bias	RMSE
25	5.04	0.04	0.05	200	5.04	0.04	0.04
	0.80	0.30	0.30		0.79	0.29	0.29
	1.55	0.05	0.05		1.54	0.04	0.04
	0.54	0.04	0.05		0.56	0.06	0.06
	1.70	0.10	0.11		1.64	0.04	0.04
50	5.04	0.04	0.04	500	5.04	0.04	0.04
	0.79	0.29	0.29		0.79	0.29	0.29
	1.54	0.04	0.04		1.54	0.04	0.04
	0.55	0.05	0.05		0.56	0.06	0.06
	1.68	0.08	0.09		1.64	0.04	0.04
100	5.04	0.04	0.04				
	0.79	0.29	0.29				
	1.54	0.06	0.04				
	0.56	0.06	0.06				
	1.65	0.05	0.06				

Table 3

Simulation Results for Case III (1.6, 2.5, 0.5, 0.5, 4)

Sample Size	Estimates	Bias	RMSE	Sample Size	Estimates	Bias	RMSE
25	1.59	-0.01	0.01	200	1.59	-0.01	0.01
	2.89	0.39	0.39		2.89	0.39	0.39
	0.51	0.01	0.01		0.51	0.01	0.01
	0.50	0.00	0.00		0.50	0.00	0.00
	4.02	0.02	0.02		4.01	0.01	0.01
50	1.594	-0.01	0.01	500	1.59	-0.01	0.01
	2.89	0.39	0.39		2.89	0.39	0.39
	0.51	0.01	0.01		0.51	0.01	0.01
	0.50	0.00	0.00		0.50	0.00	0.00
	4.01	0.01	0.02		4.01	0.01	0.01
100	1.59	-0.01	0.01				
	2.89	0.39	0.39				
	0.51	0.01	0.01				
	0.50	0.00	0.00				
	4.01	0.01	0.01				

Table 1 to Table 3 can be summarized as follows:

i. The estimates are relatively good as they are close to the actual parameters for all the cases.

ii. For Table 1, the Biases and RMSEs for the $\hat{\alpha}, \hat{\gamma},$ and $\hat{\lambda}$ drastically decrease and approach zero as the sample sizes increase, whereas, $\hat{\eta}$ and $\hat{\kappa}$ slightly misbehave.

iii. For Table 2, the Biases and RMSEs for $\hat{\alpha}, \hat{\gamma}, \hat{\lambda},$ and $\hat{\kappa}$ drastically decrease and approach zero as the sample sizes increase. However, $\hat{\eta}$ slightly misbehaves.

iv. For Table 3, the Biases and RMSEs for $\hat{\eta}$ and $\hat{\kappa}$ drastically decrease and approach zero as the sample sizes increase. However, $\hat{\alpha}, \hat{\gamma},$ and $\hat{\lambda}$ slightly misbehave.

Application

The following data set was first published by Badar and Priest (1982). It reveals the GPa Strength Readings of 69 individual carbon fibers under stress at 20 mm gauge measurements. The gauge lengths at which the data were gathered varied.

0.312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.977, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301.

ILWWD alongside the Extended Odd Frechet Weibull Distribution (EOFWD) by Nasiru (2018), the Exponentiated Weibull Distribution by Pal et al. (2006), and the Weibull Distribution were fitted to the data. To assess the flexibility of the ILWWD, several statistical metrics were applied, including the Hannan-Quinn Information Criterion (HQIC), the Bayesian Information Criterion (BIC), the Akaike Information Coon Criterion (AICc), and the Akaike Information Criteria (AIC). It is considered better to have a distribution with lower AIC, AICc, BIC, and HQIC values. Additional indicators were also employed, like the Anderson-Darling test (A), the Cramer-Von Mises Criterion (W), and the Kalmogorov-Smirnov test (K-S). ILWWD is the best distribution with lower values of the criteria and a larger P-value, as indicated in Table 4. Figure 5 also indicates the superiority of the ILWWD over the other distributions in terms of fitting. Table 5 presents the MLEs with their standard errors in parenthesis. The total time on the test (ttt) plot of the data is presented in Figure 6. This indicates that the strength reading data has an increasing hazard rate shape.

Table 4

The Goodness-of-fit Statistics for the fitted ILWWD and other distributions for the Strength Readings Dataset

Distributions	W	A	KS	P-value	AIC	BIC	HQIC	CAIC	ll
ILWWD	**0.05**	**0.39**	**0.12**	**0.79**	**4.87**	**11.53**	**6.91**	**7.59**	**-2.56**
EOFWD	0.09	0.57	0.17	0.41	6.56	11.88	8.19	8.29	-0.72
EWD	0.24	1.53	0.31	0.01	24.80	28.80	26.08	25.80	9.40
WD	0.13	0.94	0.16	0.49	10.85	13.52	11.67	11.33	3.43

Table 5

The MLEs and Standard Errors (S.E) for ILWWD and Others for the Strength Readings Dataset

Distribution	Estimates (S.E)
ILWWD $(\hat{\alpha},\hat{\gamma},\hat{\lambda},\hat{\eta},\hat{\kappa})$	5.39(2.07), 0.37(0.14), 2.99(2.38), 0.79(0.13), 0.77(0.12)
EOFWD $(\hat{\alpha},\hat{\gamma},\hat{\lambda},\hat{\eta})$	1.41(0.04), 8.43(0.00) 3.96(0.01), 0.09(0.13)
EWD $(\hat{\alpha},\hat{\gamma},\hat{\lambda})$	1.24(1.02), 1.84(0.34), 1.72(2.01)
WD $(\hat{\alpha},\hat{\gamma})$	4.25(0.21), 1.07(1.02)

Figure 5

The Fitted Distributions with the Data

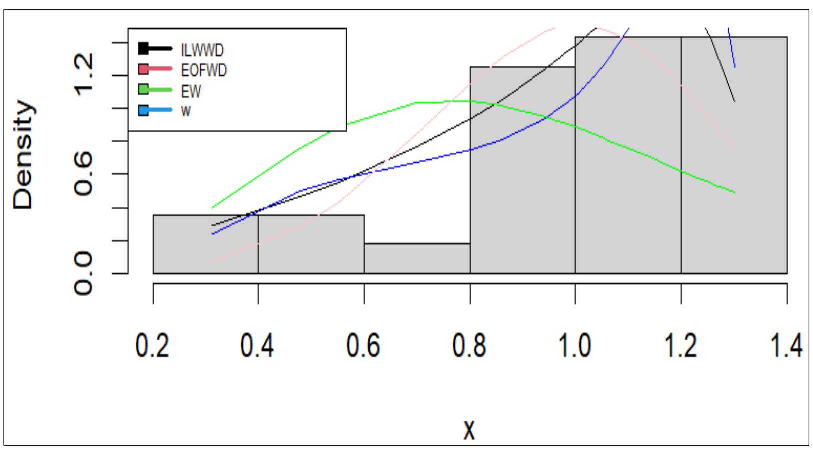
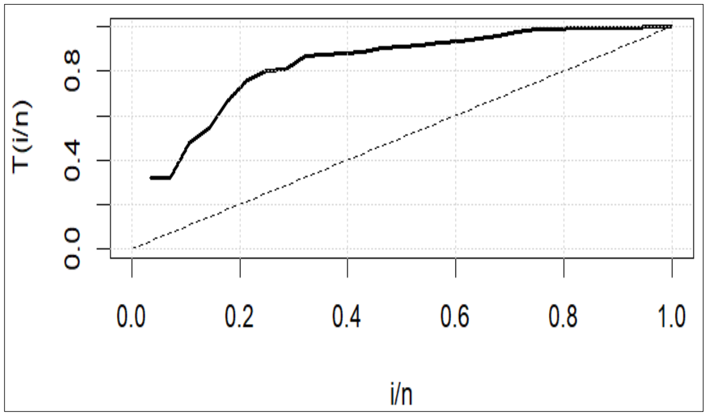


Figure 6

The ttt plot for the data



Survival Regression for the ILWWD

This data is for survival times for 33 leukemia patients, as reported by Feigl and Zelen (1965) and Lawless (2011), as presented in Table 6. The survival times are in weeks y_i from diagnosis, and there are two covariates: a binary variate AG that indicates a positive (AG = 1) or negative (AG = 2) test related to white blood cell characteristics x_1 and white blood cell count (WBC) at diagnosis as x_2. And a censoring 1= death, 0 = alive, as d. The Log-ILWWD model given in Equation (24) as fitted alongside the Log-Generalized Exponentiated Frechet-Weibull Distribution (Log-GEFWD) by Klakattawi et al. (2023) and Log-Topp Leone Odd Log-logistic Weibull Distribution (Log-TLOLLW) by Brito et al. (2017). Table 6 reports the estimated parameter values, standard errors, the negative value of log-likelihood, the BIC, and the AIC for the regression model fitted by MLE. Considering that the Log-ILWWD regression model has the lowest AIC and BIC statistics values among the two compared regression models (Log-GEFWD and Log-TLOLLWD), as indicated in Table 7, we may infer that the Log-ILWWD regression model yields a superior fit.

Table 6

The Leukemia Data

Time	d	x1	x2
65	0	2.3	1
140	1	0.75	1
100	0	4.3	1
134	0	2.6	1
16	0	6	1
106	1	10.5	1
121	0	10	1
4	0	17	1
39	0	5.4	1
121	1	7	1
56	0	9.4	1
26	0	32	1
22	0	35	1
1	0	100	1
1	0	100	1
5	0	52	1
65	0	100	1
56	0	4.4	2
65	0	3	2
17	0	4	2
7	0	1.5	2
16	0	9	2
22	0	5.3	2
3	0	10	2
4	0	19	2
2	0	27	2
3	0	28	2
8	0	31	2
4	0	26	2
3	0	21	2
30	0	79	2
4	0	100	2
43	0	100	2

Table 7

The Survival Regression Results

Parameter	**Log-ILWWD**			Log-TLOLLWD			Log-GEFWD		
	Estimates	**Stand. Err**	**P- value**	Estimates	Stand. Err	P-value	Estimates	Stand. Err	P-value
α	**4.20**	**0.01**	**0.00**	7.09	4.93	0.15	1.95	1.58	0.22
$\boldsymbol{\lambda}$	**12.12**	**0.06**	**0.00**	3.11	1.62	0.06	13.05	11.12	0.24
γ	**25.27**	**0.01**	**0.00**	---	---	---	6.56	8.59	0.45
β	---	---	---	---	---	---	0.16	0.29	0.56
ε	---	---	---	---	---	---	9.77	21.25	0.65
σ	**13.89**	**0.00**	**0.00**	0.37	1.74	0.83	0.39	1.62	0.81
τ_0	**-11.25**	**1.00**	**0.00**	4.33	15.29	0.78	13.71	8.88	0.12
τ_1	**0.02**	**0.25**	**0.00**	0.01	0.05	0.88	-0.78	0.71	0.27
τ_{02}	**17.12**	**1.06**	**0.00**	-4.12	10.47	0.69	0.41	22.01	0.99
	AIC	**17.59**			48.78			27.80	
	BIC	**21.07**			57.76			41.27	
	-2ll	**3.59**			36.78			9.80	

CONCLUSION

In this paper, a new extension of the Weibull distribution called ILWWD is presented. That is, some of the statistical properties of the ILWWD were presented. A simulation study for three different cases indicated that the estimates of the ILWWD were consistent. ILWWD was also applied to a real-life dataset. This also indicated the significance of the ILWWD, as it outperforms the other distribution. Furthermore, the Log-ILWWD was presented and applied to leukemia data, indicating the superiority of the Log-ILWWD as it outperforms the other distributions compared in the study.

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