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MATHEMATICAL MODELING OF VISCOUS INCOMPRESSIBLE FLUID FLOW IN TUBULAR NETWORKS: ANALYSIS OF LAMINAR AND TURBULENT REGIMES

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ABSTRACT

The article delves into the mathematical modeling of the flow of viscous, incompressible fluids through a network of tubes enclosed within an outer pipe. It comprehensively examines laminar and turbulent flow regimes, thoroughly analyzing the underlying physical phenomena. Specifically, it investigates the fluid dynamics through a configuration consisting of n tubes, each with a length L and radius r, situated within the outer tube. The article derives formulae for calculating key parameters such as the maximum velocity of fluid flow, the volumetric flow rate through the tube's cross-section, the coefficient of frictional resistance along the tube's length, and the maximum value of tangential stress.

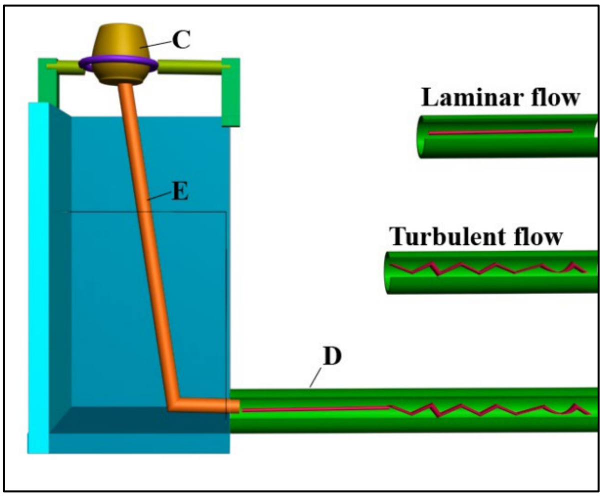
Keywords: Darcy-Weisbach, friction force, parabolic flow, Poiseuille, Reynolds number.

INTRODUCTION

Viscosity in liquids acts as a resistance to the movement of different layers of liquid sliding past each other. In simpler terms, when liquids flow in a smooth, layered manner, viscosity causes internal friction. This friction is quantified by the tangential stresses at the boundaries between these layers or by the tangential forces per unit area. Essentially, in laminar flows, adjacent concentric layers of liquid move in a way that the liquid's velocity aligns with the main axis. This type of fluid movement is referred to as laminar flow (Abutaliev & Normurodov, 2011; Drazin, 2005 & Thomas, 1953). The behavior of real liquids often deviates significantly from laminar flows due to a distinctive characteristic known as turbulence. In practical situations, such as fluid flow through pipes, channels, and boundary layers, as the Reynolds number increases, there is a noticeable shift from laminar flow to turbulent flow. This transformation from laminar to turbulent flow is referred to as turbulence and holds significant significance in the field of hydrodynamics. Initially, this transition was observed in the context of fluid currents within straight pipes and channels. In a straight pipe with a smooth interior surface and a constant cross-sectional area, individual particles within the liquid tend to follow linear paths, especially when the Reynolds numbers are low. Due to the inherent viscosity of the liquid, particles close to the pipe wall move more slowly in comparison to those situated farther from the wall. As a consequence of this behavior, the flow takes on an organized pattern characterized by distinct layers of liquid moving relative to each other. However, when Reynolds numbers significantly increase, the flow undergoes a transition into a chaotic state known as turbulent flow. In this turbulent state, there is a marked tendency for the liquid to mix vigorously. This mixing can be visually demonstrated by introducing dye or paint into the flowing liquid within the pipe. The pioneering work conducted by Reynolds (1883) encompassed these practical experiments in which he introduced paint into the liquid to investigate this phenomenon (Abutaliev & Normurodov, 2011). As the flow transitions from laminar to turbulent, we observe a distinct change in the behavior of paint within the pipe. In laminar flow, the paint moves in a well-defined stream, while in turbulent flow, it diffuses across the entire pipe, coloring the bulk of the liquid. This phenomenon highlights that in turbulent flow, there is a transverse or perpendicular movement to the pipe's axis in addition to the flow along the pipe's axis. This lateral movement is responsible for thoroughly mixing the dye throughout the fluid (see, for instance, Abutaliev & Normurodov, 2011; Drazin, 2005; Goldshtik & Shtern, 1977 – Goldshtik & Sapozhnikov, 1968; and Gorshkov 2015 - Thomas, 1953). The apparatus used for conducting this experiment is depicted in Figure 1 below.

Figure 1

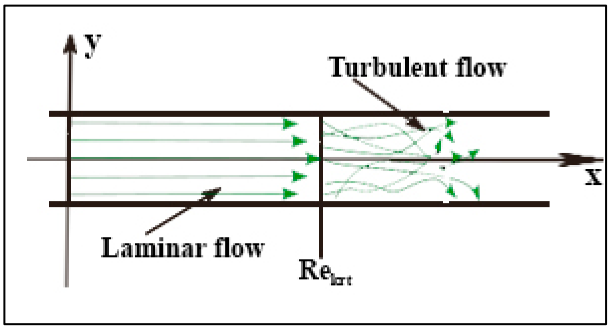
The experiment in which paint is introduced into the liquid.



The experiment commences by flowing a liquid through the pipe labelled "D" at low speeds. Simultaneously, paint is supplied from tank "C" through tube "E." This results in a specific observation: the painted stream maintains a straight horizontal line while the remaining flowing fluid remains uncolored. Consequently, in this scenario, the particles within the painted stream do not intermingle with the rest of the liquid, indicating that the flow within pipe "D" is in a laminar mode. As the velocity within pipe "D" gradually increases, there comes the point at which the painted stream vanishes, and the entire moving liquid becomes uniformly colored. This change signifies that the liquid particles within the flow have mixed, indicating a transition to a turbulent regime within pipe "D." When the incompressible viscous fluid moves starting at the same value of the Reynolds number $\mathrm{Re} = \rho UL / \mu$, the laminar flow passes into a turbulent one, the same value of the Reynolds number is called the critical Reynolds number, where ρ -density, μ - viscosity of the liquid, U - the maximum velocity of the main flow, L - the characteristic scale of the length.

Figure 2

The transition from laminar flow to turbulent



From Figure 2, it is observed that at $\mathrm{Re} < \mathrm{Re}_{krt}$, laminar flow and $\mathrm{Re}_{krt} < \mathrm{Re}$, the flow goes into turbulent mode.

STATEMENT OF THE PROBLEM

In a study by Bakhvalov (1969), details are provided regarding the forces at play in the flow within a cylindrical tube. We are specifically examining the flow of a fluid through a straight, circular pipe of constant diameter, which contains an array of smaller tubes within it, each with a length (denoted as L) and a radius (denoted as r). In real liquids, the fluid adheres to the inner walls of the tube, transferring shear stress to the surface of the streamlined fluid. This phenomenon gives rise to what is known as internal friction, with viscosity being the manifestation of this property in liquids. Viscosity is a characteristic property of both gases and liquids, representing their resistance to motion induced by external forces. The presence of tangential stresses and the adhesion of liquids to solid walls result in significant distinctions between real and ideal liquids. Our current investigation focuses on the motion of liquids within a pipe that contains these smaller tubes, each of identical length and radius. When accounting for viscosity at the tube walls, the velocity of the fluid is zero at the tube's entrance, reaching its maximum value at the midpoint of the tube. At a sufficiently distant location from the tube's entrance, the distribution of flow velocity becomes independent of the radial coordinate. Fluid motion within the pipe is driven by a pressure gradient along the pipe's axis. However, within each cross-section perpendicular to the pipe axis, the pressure remains relatively constant. The acceleration and deceleration of individual fluid elements are a result of this pressure gradient, coupled with the decelerating effect of shear stress induced by friction (see, for instance, Abutaliev & Normurodov, 2011; Drazin, 2005; Goldshtik & Shtern, 1977 – Goldshtik & Sapozhnikov, 1968; and Gorshkov 2015 - Thomas, 1953). The pressure p is assumed to be constant. That is, it is assumed that over the section of the tube $p_0, p_l = const$, Brown, 1961. In the direction of the main axis, a pressure force $p_0 n\pi y^2$ and $p_l n\pi y^2$ is applied to the inlet and outlet bases of the tube, respectively, as well as the tangential force $2\pi nyL\tau$ acting on the lateral surface of the cylinder, acting on the tubes. Hence, it is required to determine the maximum flow velocity in the tube, the volume of fluid flowing through the cross-section of the tube, the coefficient of tube resistance to friction along the flow length, as well as the maximum value of the tangential stress.

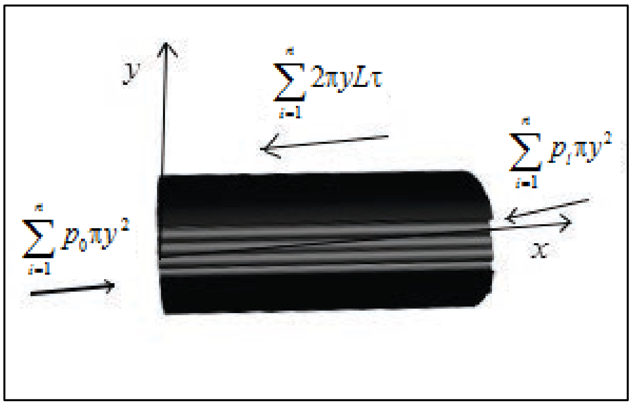
SOLUTION OF THE PROBLEM

Equating the forces acting fluid in the tube, we obtain an equilibrium condition in the direction of motion in the Equation 1.

$$\sum_{i=1}^{n} p_0 \pi y^2 = \sum_{i=1}^{n} p_l \pi y^2 + \sum_{i=1}^{n} 2\pi y L\tau \cdot \tag{1}$$

Figure 3

Several n tubes are located in the tubes.



The projection of the internal friction force is taken with a plus sign since the velocity gradient is negative (the velocity of the layer decreases with increasing radius r). From Equation (1), we determine the tangent stress, τ

$$\tau = \frac{p_0 - p_l}{L} \cdot \frac{y}{2}. \tag{2}$$

In this case, the flow velocity u decreases with increasing coordinate y and is zero at $y = r$. Therefore, on the basis of the law of friction is $\tau = \mu \frac{du}{dy}$. According to Hooke's law of friction, $\tau = -\mu \frac{du}{dy}$. By substituting this expression in Equation (2), we obtain:

$$-\mu \frac{du}{dy} = \frac{p_0 - p_l}{L} \cdot \frac{y}{2}.$$

We can easily see that

$$\frac{du}{dy} = -\frac{p_0 - p_l}{\mu L} \cdot \frac{y}{2}, \tag{3}$$

by integrating Equation (3), we obtain

$$u(y) = -\frac{p_0 - p_l}{4\mu L} y^2 + C. \tag{4}$$

In order to determine the constant C of Equation (4), we use the initial condition $u(r) = 0$, and hence we have

$$u(r) = -\frac{p_0 - p_l}{4\mu L} r^2 + C = 0.$$

Solving the previous Equation for C, we obtain

$$C = \frac{p_0 - p_l}{4\mu L} r^2. \tag{5}$$

Substituting the value of the constant C from Equations (5) to (4), we obtain

$$u(y) = -\frac{p_0 - p_l}{4\mu L} y^2 + \frac{p_0 - p_l}{4\mu L} r^2.$$

After simplifying this Equation, we obtain an equation for the flow rate as follows

$$u(y) = \frac{p_0 - p_l}{4\mu L} (r^2 - y^2). \tag{6}$$

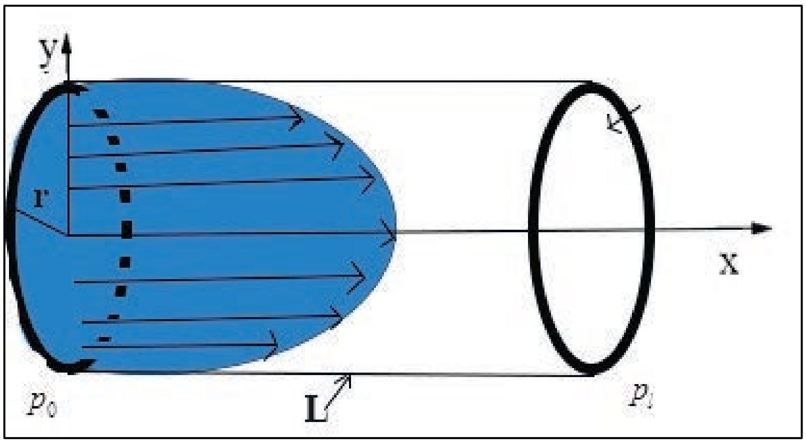
Thus, we have a parabolic velocity distribution along the radius of the pipe, as displayed in Figure 3. The greatest value of speed is

$$u_{\max} = \frac{p_1 - p_2}{4\mu L} r^2, \tag{7}$$

in the middle of the pipe, where $y = 0$.

Figure 4

Fluid flow rate for one tube.



The total amount of Q of liquid flowing through the pipe section (fluid flow) is defined as the volume of the paraboloid of rotation, as illustrated in Figure 4, and acreage will be defined in the following way. From Equations (6) and (7), we get the following

$$u(y) = \frac{p_0 - p_l}{4\mu L} r^2 (\frac{r^2 - y^2}{r^2}) = u_{\max} \left(1 - \frac{y^2}{r^2}\right). \quad (8)$$

The calculation of the total liquid flow through a tube with a circular cross-section, based on the Gagen-Poiseuille formula, is conducted as follows:

$$Q = \int_0^r u(y) 2\pi y dy = 2\pi u_{\max} \int_0^r \left(y - \frac{y^3}{r^2} \right) dy == 2\pi u_{\max} \left[\frac{y^2}{2} - \frac{y^4}{4r^2} \right]_0^r = r^2 \pi u_{\max} .$$

By substituting Equation (7) into the last relation, we have

$$Q = 2\pi \cdot \frac{p_0 - p_l}{4\mu L} \cdot r^2 \cdot \frac{r^2}{4} = \frac{\pi (p_0 - p_l) r^4}{8\mu L} . \quad (9)$$

Dividing both sides of the last expression by $2\pi r^2$ and letting

$$\overline{u} = \frac{Q}{\pi r^2} .$$

Equation (9) can be written as follows

$$\overline{u} = \frac{(p_0 - p_l) r^2}{8\mu L} . \quad (10)$$

Comparing the function $\bar{u}(y)$ with the maximum speed $u_{\max}$ that was determined in Equation (7), one can see that $\bar{u}(y)=\frac{1}{2}u_{\max}$. This implies that the average velocity of the laminar flow within the tube is equal to half of the maximum velocity, as illustrated in Figure 5. Determine the pressure difference (p_0-p_l) as follows:

$$p_0-p_l=\frac{8\mu L\bar{u}}{r^2}.$$

From this, it follows that

$$p_0-p_l=\frac{8\mu L\bar{u}}{r^2}=\frac{32\mu\bar{u}}{2r}\cdot\frac{L}{2r}=\frac{32\mu\bar{u}}{D}\cdot(\frac{L}{D}), \tag{11}$$

where $D=2r$ is the diameter of the tube. The pressure loss along the flow length is determined by the Darcy-Weisbach equation:

$$p_0-p_l=\sum_{i=1}^{n}\frac{\lambda_i}{2}\rho\bar{u}^2(\frac{L}{D}), \tag{12}$$

where, λ - is the hydraulic loss ratio along the length of the pipe or the resistance coefficient of the pipe. From the previous equation, we have:

$$\lambda=\frac{(p_1-p_2)}{\frac{1}{2}\rho\bar{u}^2}\cdot(\frac{D}{L}). \tag{13}$$

Substituting p_0-p_l the value of Equation (11) in Equation (13), we obtain the resistance coefficient of the pipe in the following:

$$\lambda=\frac{32\mu\bar{u}}{D}\cdot(\frac{L}{D})\cdot\frac{2}{n\rho\bar{u}^2}\cdot(\frac{D}{L})=\frac{64\mu}{n\rho\bar{u}D}.$$

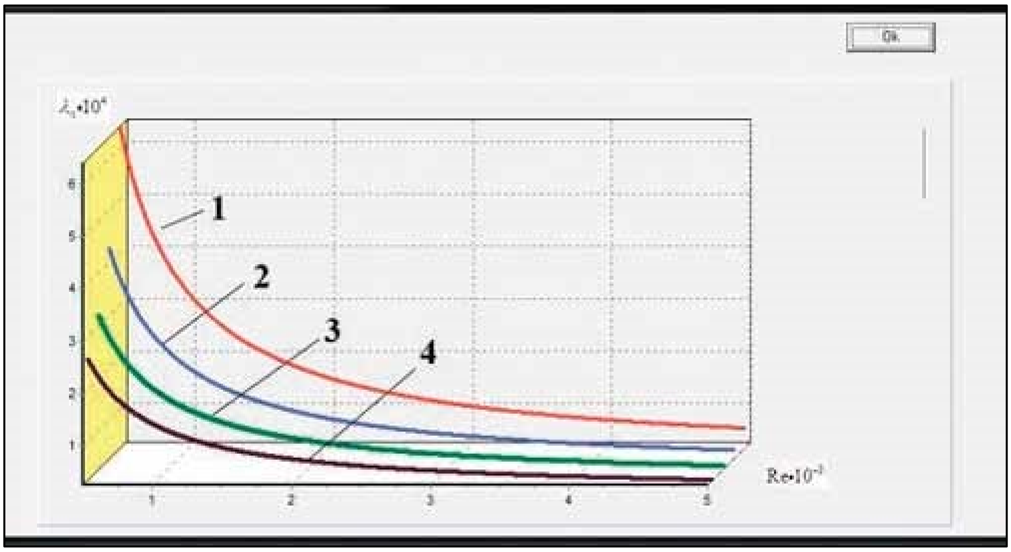
It implies that

$$\lambda=\frac{64}{n\,\mathrm{Re}}, \tag{14}$$

where $\mathrm{Re}=\frac{\rho\bar{u}D}{\mu}$ is the Reynolds number. According to Equation (14), to calculate the resistance coefficient, we present the results of calculations for various numbers of tube bundles n, as displayed in Figure 5 below.

Figure 5

The dependence of the resistance coefficient for smooth tubes on the number of tubes n and the Reynolds number Re: 1) n = 200, 2) n = 300, 3) n = 400, 4) n = 500.



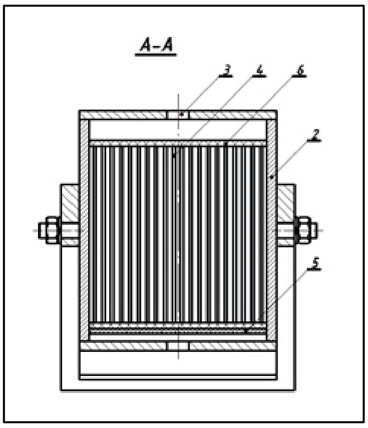
For n smooth tubes, Figure 5 displays the results illustrating the dependence of the resistance coefficient λ_n on the Reynolds number Re.

ANALYZING THE RESULTS

A comparison of the obtained results reveals that for all values of the Reynolds number, the theoretical Equation (14) holds. In computational experiments, the following range of variation of the characteristic parameters Re and λ_n: $\mathrm{Re} = 500 \div 5000$, $\lambda_n = 0.0001 \div 0.0007$ was considered. From Figure 5, it is observed that as the number of tubes n increases, the resistance coefficient decreases. Thus, it is suggested that the motion of incompressible viscous flows in channels, pipes and the boundary layer can be laminar and turbulent, and the physical meaning of the appearance of these modes is indicated. For the fluid flowing through the tube n inside the tube, the formulas for calculating the maximum velocity of the fluid volume flowing through the cross-section of the tube, as well as the coefficient of tube resistance to friction along the length of the flow, are derived. Based on the above results, it is possible to create a device that allows the regulation of water flows. The device can be applied to the production process to regulate the flow of water fountains. The proposed device allows one to make the fountains appear beautiful, enriching them with different colors. The main purpose of the device is to maintain the flow of water in the form of a uniform laminar flow using its working bodies. The device can be explained by the following diagrams, which present its working elements. Figure 6 illustrates the internal section of the device, where the base legs hold the device at an acute angle, and the device is firmly attached to it. It contains two tube-shaped bodies, three mounted vertically from bottom to top, a cylindrical section with inlet and outlet holes, and four small-diameter tubes that serve to convert the tubular flow into a laminar flow. This includes a distributor that serves to distribute the flow evenly to the small-diameter tubes by reducing the flow rate to 5 and a grid holding the 6-diameter tubes, which in turn serves the orderly movement of the flow.

Figure 6

Internal view of the device.



The device that ensures the flow laminar has the following appearance: the flow from inlet 3 is directed to flow 5, which reduces the flow rate. It distributes the flow evenly in small diameter pipes. Meanwhile, the flow passing through pipes with a diameter of less than 4 comes in a laminar view and is blown in a laminar view through three outlet holes. The experimental study discovered that the optimal length for a set of tubes placed inside a tube was 12 to 16 cm and indicated a decrease in the resistance coefficient as the tube set increased. This results in the creation of a device to convert chaotic currents into laminar currents.

CONCLUSION

This study examines the movement of viscous fluids in both laminar and turbulent regimes within incompressible channels, pipes, and boundary layers. It also explores the physical mechanisms responsible for the transition between these flow regimes. The process of adding color to a moving fluid, as recommended by Reynolds, is demonstrated through experimentation. This research involves determining the maximum liquid flow velocity around a cylinder placed within a pipe and calculating the volume of liquid passing through the pipe's cross-sectional area. It also derives formulas for calculating the pipe's friction resistance coefficient based on flow length. The study includes the presentation of pipe resistance coefficients, similar to the studies of Reynolds (1883), Abutaliev and Normurodov (2011) and Brown (1961), alongside experimental results obtained using these formulas. Additionally, it explores the range of variation in critical Reynolds numbers for Poiseuille and Blasius flows. The findings also extend to Poiseuille flow within a flat channel Gorshkov 2015 - Thomas, 1953, highlighting their proportional relationships.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

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