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Discovering the Fascinating Pattern in the Geometric Representation of Non-intersecting Chords on the Circle within the Motzkin Sequence

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***Abstract*—This study explores the Motzkin sequence and introduces a new sequence, the Wing sequence, derived from a novel geometric representation of directed paths. The main objective is to investigate patterns in the geometric representation of the Motzkin sequence and analyze these representations to construct the Wing sequence. The Motzkin sequence, which counts the number of ways to draw non-intersecting chords between n points on a circle serves as the basis for this study. The geometric representation of these chords, particularly for $n = 1, 2, 3$ and 4, is analyzed to reveal patterns and properties that could be modified to generate a new sequence. Based on the analysis, we tried some iterations for $n = 3$ and $n = 4$ to develop initial ideas for constructing the Wing sequence. The technique involved modifying the geometric representation of the Motzkin sequence to derive the Wing sequence. This process included transforming the non-intersecting chords into a circular representation into a linear arrangement. We then removed any representations without chords and only considered those with chords. Next, we transformed the chords into directed paths. Since these directed paths only connected two points, we combined them to form directed paths that passed through all points at least once. These results identified a pattern between the first and second points, leading to the Fist-Second-Third-Points (FSTP) technique for constructing the Wing sequence. The main findings include deriving a general formula for the Wing sequence, establishing a recurrence relation, and constructing a generating function. These results highlight the applicability of the geometric representation technique in discovering new sequences and enhancing geometric representation techniques in mathematical research. The techniques developed in this study can be applied to other geometric problems, offering researchers a more intuitive understanding of structures. Further exploration of these results may open up various applications for combinatorial structures, such as network routing issues, project scheduling, and key generation involving data security.**

***Keywords*—Motzkin sequence; geometric representation; generating function; recurrence relation.**

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I. INTRODUCTION

The Motzkin sequence, introduced by Theodore Motzkin in 1948, is represented as M_n for $n = 0, 1, 2, \ldots$. The sequence begins as follows: 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, and so on. Each term in the sequence represents the number of ways to draw non-intersecting chords between n points on a circle [1]. Table 1 provides an example of $_4 = 9$. Figure 1 illustrates that the Motzkin sequence can be derived from the geometric representation of non-intersecting chords on a circle. To be more specific, $M_4 = 9$ indicates that there are nine distinct ways to draw non-intersecting chords connecting 4 points placed evenly around the circumference of a circle. Each configuration represents a unique arrangement where the chords do not cross each other inside the circle, capturing all possible ways to connect the points without intersections.

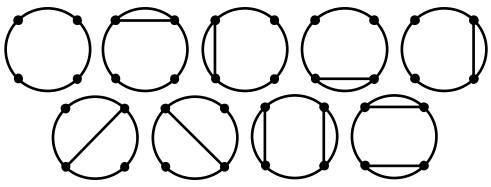


Fig. 1 Nine ways of drawing non-intersecting chords on a circle of 4 points [2]

This representation highlights the Motzkin sequence's combinatorial nature and underscores its connection to geometric arrangements. It offers insight into the sequence's structure and its applications in various fields of mathematics.

The benefit of using geometric representation for graphs is that it offers a visual way to illustrate sequences. For instance, consider the representation of a rectangle where the Motzkin numbers are used to depict the area added to its semi-perimeter, as well as the area subtracted by the semi-perimeter [3].

Despite the Motzkin sequence, extensive research has focused on its geometric representation. For instance, the geometric representation of a polygon triangulation generated the Catalan sequence [4], [5], [6], [7], and the lattice paths on rectangular grids produced the Dellanoy sequence [8], [9], [10]. As a result, the Riordan sequence emerged from the geometric representation of the Motzkin sequence within the realm of polyhedral combinatorics [11], [12], prompting researchers to explore new sequences.

Exploring new sequences can offer significant contributions to various fields of mathematics with intriguing applications. For example, the Motzkin sequence has numerous uses in combinatorics, such as counting the number of ways to arrange parentheses in valid sequences and addressing problems related to lattice paths [12]. In geometry, the Motzkin sequence is related to the number of ways to connect points in a plane without crossing lines. This has applications in geometric graph theory and planar graphs [13]. The Motzkin sequence also appears in number theory, where it is linked to problems involving partitions and integer sequences [14], [15]. Other than that, it has applications in probabilistic models where non-crossing events are considered, such as specific random walks or stochastic processes [16]. In algorithm design and analysis, the Motzkin sequence can contribute to counting specific configurations or structures in computational problems [17]. Overall, the Motzkin sequence is a versatile sequence that can solve problems in diverse mathematical and applied contexts [1], [18], [19], [20].

While exploring the pattern of the geometric representation of the Motzkin sequence, we identified some interesting patterns that can be directly modified to obtain a new sequence. We have introduced a new sequence, namely the Wing sequence, W_n, which is inspired by the geometric representation of the Motzkin sequence. The modification technique from [21] is employed to construct the first five enumerations of W_n as presented in [22]. Hence, this paper illustrates the fourth enumeration by modifying the geometric representation of non-intersecting chords on a circle within the Motzkin sequence. Based on this pattern, we also present the results for the sixth enumeration. The visualization of the sixth enumeration of W_n provided in Section 4, offers a more transparent representation compared to the method described in [22]. Indeed, the visualization technique has been shown to enhance people's thinking skills and develop their visual abilities [23], [24], [25], [26], [27], [28], which explains the amendments made in constructing the sixth enumeration of W_n from the Motzkin sequence.

This paper aims to uncover the hidden patterns in the geometric representation of directed paths formed by non-intersecting chords on a circle within the Motzkin sequence. This concept is the foundation for introducing innovative methods to generate other sequences.

II. Materials and Method

A. Preliminary Definitions

Below are some preliminary definitions relevant to this paper.

Definition 1 [2]. A non-intersecting chord on a circle is a straight line joining any two points on the circumference of a circle, with the condition that the chord is not connected to another chord.

Definition 2 [29]. A geometric representation of graph is defined as a drawing of any objects in a plane consisting of points and curves, or lines connecting them.

Definition 3 [22]. A Wing sequence denoted by $\boldsymbol{W_n}$ consists of elements (or terms) where each term is the number of unique ways of drawing distinct directed paths among $\boldsymbol{n}$ points such that each point is necessarily passed by at least once.

Definition 4 [22]. For any $n \geq 1$, STP is defined as the starting point such that $STP = 1$.

Definition 5 [22]. For any $n \geq 2$, SCP is defined as the second point such that STP is defined as the starting point such that $2 \leq SCP \leq n$.

Definition 6 [22]. $Class\ i[SSP(1, i+1)]$ for $1 \leq i \leq n-1$, where SSP represents the "Starting and Second point" describes the set of STP and SCP of the directed paths in each class.

Definition 7 [30]. Let $(a_n)_{n\geq 0}$ be a sequence of numbers. The generating function associated with the sequence is the series $A(x) = \sum_{n\geq 0}^{\infty} a_n\, x^n$ or can be expanded to generate series $a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$.

Definition 8 [30]. A recurrence relation or simply recurrence for a sequence $\langle a \rangle$ is an expression for a_n in terms of n and the values $a_0, \ldots, a_n$. If the expression for a_n depends only on n and $a_{n-k}, \ldots, a_{n-1}$ (and is used only when $n \geq k$), then the recurrence is of order k.

These definitions will be used in the following sections to discuss the geometric representations of non-intersecting chords on a circle and the geometric representations of directed paths among *n* points.

B. The Geometric Representation of the Motzkin Sequence

Each term of the Motzkin sequence represents the number of different ways that non-intersecting chords can be drawn between n points on a circle. The number can be defined as follows:

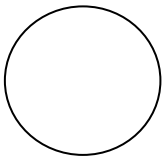
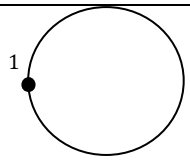
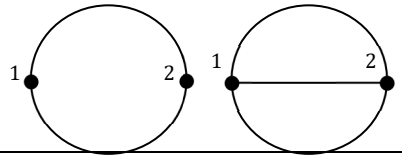
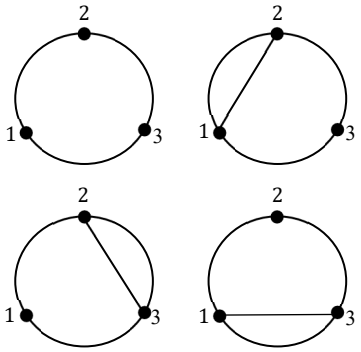
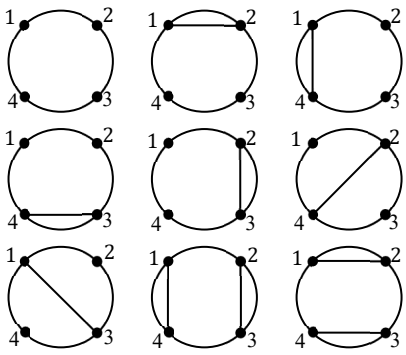
$$M_n = \frac{2n+1}{n+2} M_{n-1} + \frac{3n-3}{n+2} M_{n-2} \qquad (1)$$

for $n \geq 2$ with initial conditions $M_0 = M_1 = 1$ [2].

The distinct ways of drawing non-intersecting chords between *n* points on a circle are presented as a geometric representation in Table 1 for $n = 0, 1, 2, 3$ and 4.

TABLE I

GEOMETRIC REPRESENTATION OF NON-INTERSECTING CHORDS ON A CIRCLE DESCRIBING THE nTH TERM OF THE MOTZKIN SEQUENCE

n	Non-intersecting chords on a circle	Description	M_n
0		When no points are present, i.e. $n = 0$, there are no possible chords to draw since a chord requires at least two points on the circle. With zero chords, there is indeed just one possible geometric representation: having an empty circle with no chords at all. This is considered the trivial case. Therefore, there is **1 way** to draw non-intersecting chords with **no points** on a circle.	$M_0 = 1$
1		If there is only 1 point on the circle, i.e. $n = 1$, there are no possible chords to draw since a chord requires two distinct points to connect. Thus, the only possible geometric representation is the circle with 1 point and no chords. Therefore, there is **1 way** to draw non-intersecting chords with **1 point** on a circle.	$M_1 = 1$
2		There are **2 ways** to draw non-intersecting chords with **2 points** on a circle. One way is to draw the chord connecting the two points. The other way is to have the circle with 2 points and no chords, similar to the cases with $n = 0$ and $n = 1$.	$M_2 = 2$
3		There are **4 ways** to draw non-intersecting chords with **3 points** on a circle. One way is to have the circle with no chords. The other three ways involve drawing a chord connecting any 2 of the 3 points which lead to three distinct geometric representations.	$M_3 = 4$
4		There are **9 ways** to draw non-intersecting chords with **4 points** in a circle. One way is the circle with 4 points and no chords. The other six ways involve drawing a chord connecting any 2 of the 4 points, which leads to six distinct geometric representations. The other two ways involve drawing a chord connecting two pairs of points, which lead to two distinct geometric representations.	$M_4 = 9$

The representation in Table 1 shows that the *n*th Motzkin number is represented by the geometric representation of non-intersecting chords on a circle based on the following concepts:

- The n value is equal to the number of points.
- The M_n is the total number of ways to draw the non-intersecting chords on a circle of the *n*th term.

The patterns observed in these geometric representations of the Motzkin sequence can be further explored to design new sequences for any numbers where $n \geq 0$.

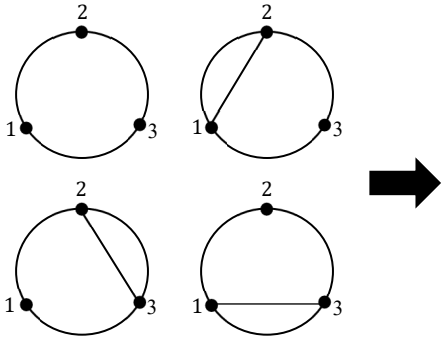
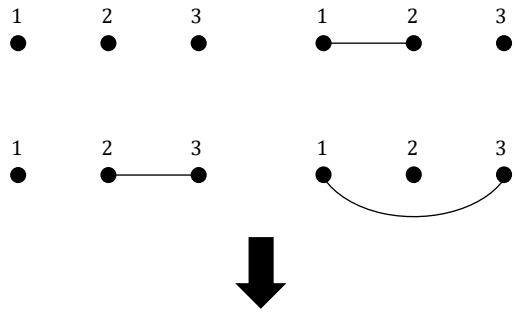
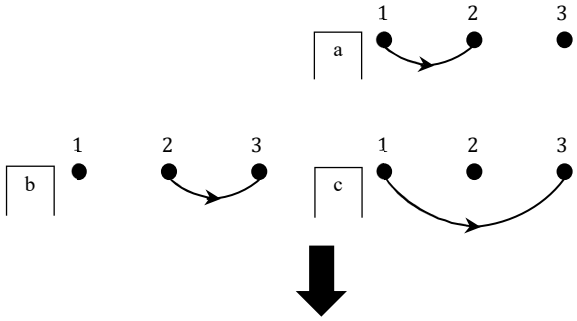
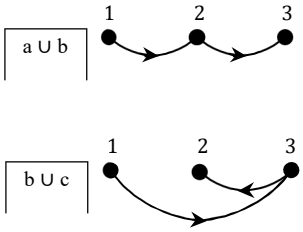
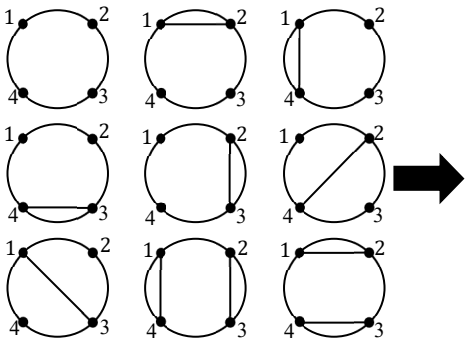
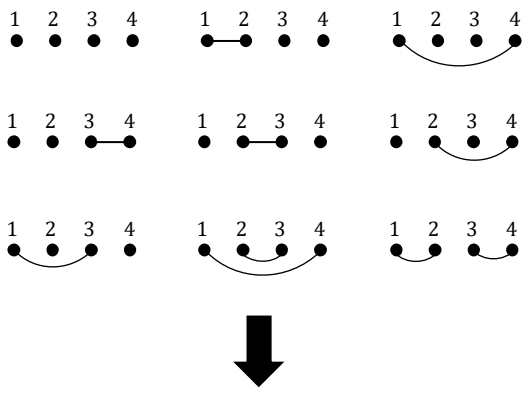
III. RESULTS AND DISCUSSION

While perceiving how non-intersecting chords on a circle produce Motzkin numbers for $n = 0, 1, 2, 3$ and 4, we can modify the technique by considering directed paths among n points. Chords on the circle can be transformed into directed paths among n points. Table 2 explores the number of ways to draw unique directed paths among n horizontal points, such that each point is passed by at least once. The first three cases are trivial, and we specifically demonstrate the results for $n = 3$ and $n = 4$.

The illustration in Table 2 becomes apparent and we are optimistic that W_n can be derived from the geometric representation of Motzkin sequence for any $n \geq 0$. Table 2 demonstrates that distinct directed paths among n points can be constructed from modified chords on a circle by employing the concept of arrow movement from one point to another. For example, the result of $n = 4$ reveals a significant pattern in the arrow movement from point 1 to the second point. Specifically, every pair 2 , close paren , open paren 1, 3 , close

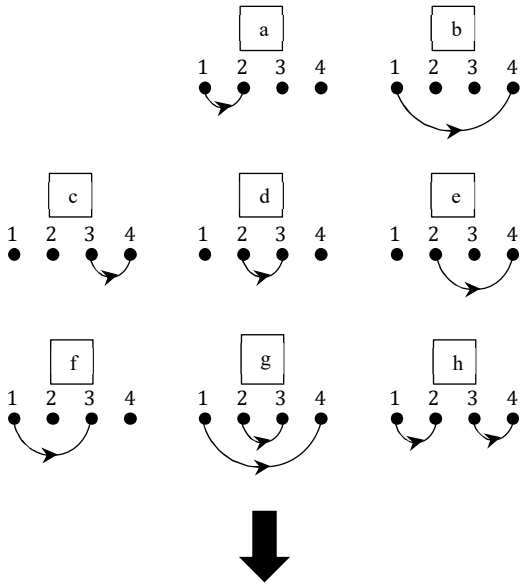
paren, and open paren 1,es the pairs $(1,2)$, $(1,3)$ and $(1,4)$. Using the same pattern, $n = 5$ has the pairs of $(1,2)$, $(1,3)$, $(1,4)$ and $(1,5)$. Similarly, $n = 6$ has the pairs of $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$ and $(1,6)$. Based on these pairs, we can obtain the distinct directed paths by moving the arrow movement to the remaining points in increasing order. Specifically, the lowest remaining point will be the third point, followed by the other points in increasing order.

TABLE II
GEOMETRIC REPRESENTATION OF DIRECTED PATHS AMONG n HORIZONTAL POINTS FOR CASES $n = 3$ AND 4.

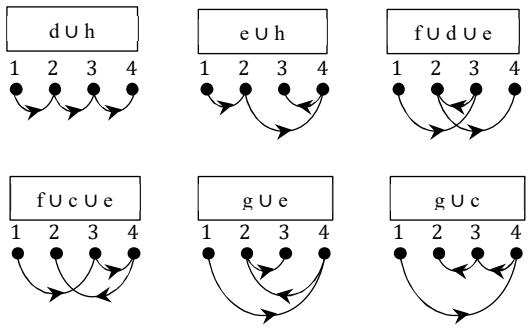
n	Non-intersecting chords on a circle	Unique directed paths among n horizontal points
3		Step 1: Remove the circle and arrange the points in a linear order.  Step 2: Remove any representations without chords. For the remaining representations with chords, transform them into directed paths. Each of these representations connects exactly two points.  Step 3: Combine the representations from step 2 to construct directed paths which passed by all points at least once. Ensure that each representation has distinct directed paths among three points and the directed path maintains continuous movement. Assume that each path begins at point 1. 
4		Step 1: Remove the circle and arrange the points in a linear order. 

n	Non-intersecting chords on a circle	Unique directed paths among n horizontal points

Step 2: Remove any representations without chords. For the remaining representations with chords, transform them into directed paths.



Step 3: Combine the representations from step 2 to construct directed paths that pass by all points at least once. Ensure that each representation has distinct directed paths among three points and that the directed path maintains continuous movement. Assume that each path begins at point 1.



We demonstrate the construction of distinct directed paths among 6 points for the pair $(1, 2)$ in Figure 2 below. Figure 2 displays four different directed paths. The blue arrow denotes the first path, the red one is the second, and the green and yellow arrows indicate the third and fourth paths, respectively.

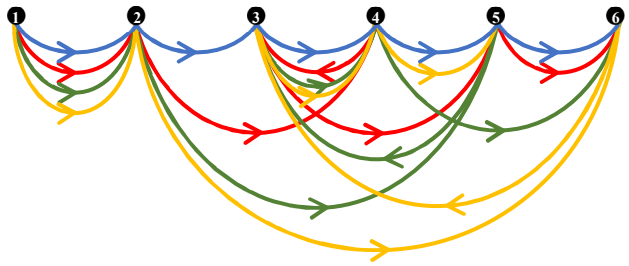


Fig. 2 $Class\ 1[SSP(1,2)]$ contains four distinct directed paths.

The first blue path connects the points $(1, 2, \mathbf{3}, 4, 5, 6)$, the second red path connects the points $(1, 2, \mathbf{4}, 3, 5, 6)$, the third green path connects the points $(1, 2, \mathbf{5}, 3, 4, 6)$, and the fourth yellow path connects the points $(1, 2, \mathbf{6}, 3, 4, 5)$. In these paths, the third point for the blue, red, green, and yellow paths are $3, 4, 5$, and 6, respectively. Since there are five classes and each class contains four paths, we will obtain a total of twenty distinct directed paths among six points.

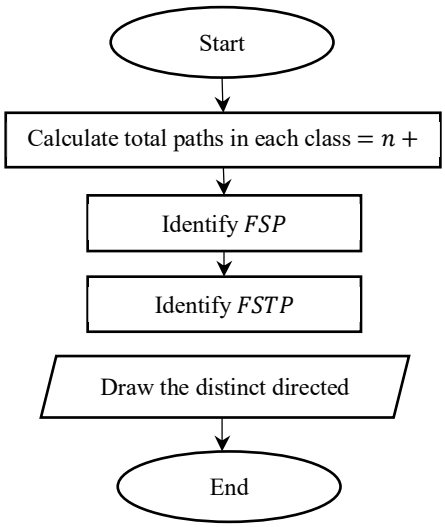


Fig. 3 Flowchart of $FSTP$ technique to construct the distinct directed path among $(n + 3)$ points for W_n.

Combining the results from [22], the first six enumerations produce the sequence $0, 1, 2, 6, 12, 20$. However, the results for 0 and 1 directed paths, which are constructed among one and two points, do not include the back-and-forth directed paths as noted in [22]. For consistency, we exclude these results and focus on the remaining ones. Thus, the initial terms of the Wing sequence are 2, 6, 12, and 20, corresponding to $W_0 = 2$, $W_1 = 6$, $W_2 = 12$, and $W_3 = 20$. These results are derived from the total number of distinct directed paths that can be constructed among $(n + 3)$ points to determine W_n. Since these paths are based on combinations of pairs from the First-Second-Point (FSP) and the third point, we propose a new technique called the FSTP (First-Second-Third-Point) technique to generate the distinct directed paths, as outlined in Figure 3. Table 3 provides the general steps to construct the distinct directed paths among $(n + 3)$ points.

TABLE III
THE FSTP TECHNIQUE TO CONSTRUCT DISTINCT DIRECTED PATH OF THE WING SEQUENCE

Step 1	**Step 2**	**Step 3**	**Step 4**	**Step 5**				
Class	**Distinct directed path, *P***	***FSP***	***FSTP***	**Points**				
$Class_1$	P_1	$(1,2)$	$(1,2,3)$	1	2	3	⋯	$n+3$
	P_2		$(1,2,4)$	1	2	4	⋯	$n+3$
	P_3		$(1,2,5)$	1	2	5	⋯	$n+3$
	⋮		⋮	⋮	⋮	⋮	⋮	⋮
	$P_{(n+1)}$		$(1,2,n+3)$	1	2	$n+3$	⋯	$n+2$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$Class_{n+2}$	$P_{(n+1)(n+1)+1}$	$(1,n$	$(1,n+3,2)$	1	$n+3$	2	⋯	$n+2$
	$P_{(n+1)(n+1)+2}$	$+3)$	$(1,n+3,3)$	1	$n+3$	3	⋯	$n+2$
	$P_{(n+1)(n+1)+3}$		$(1,n+3,4)$	1	$n+3$	4	⋯	$n+2$
	⋮		⋮	⋮	⋮	⋮	⋮	⋮
	$P_{(n+1)(n+2)}$		$(1,n+3,n+2)$	1	$n+3$	$n+2$	⋯	$n+1$

From the above FSTP technique, we have the following result.

Theorem 1. For any integer $n \geq 0$, the nth term of the Wing sequence is given by $W_n = (n + 1)(n + 2)$.

Proof.

Let $P(n)$ be the statement $W_n = (n + 1)(n + 2)$. Then, we have $P(0)$: $W_0 = (0 + 1)(0 + 2) = 2$ is true. Now, assume that the statement $P(k)$: $W_k = (k + 1)(k + 2) = k^2 + 3k + 2$ is true. Now, $P(k) \Rightarrow P(k + 1)$ where $W_{k+1} = (k + 1 + 1)(k + 1 + 2) = k^2 + 5k + 6$. Then, $W_{k+1} = W_k + 2k + 4 = (k + 1)(k + 2) + 2k + 4 = k^2 + 3k + 2 + 2k + 4 = k^2 + 5k + 6$. We have shown that $P(0)$ holds and $P(k) \Rightarrow P(k + 1)$. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all cases where $n \geq 0$.

By Theorem 1, the sequence of W_n can be written as:

$$2, 6, 12, 20, 30, \ldots, 72, 90, \ldots. \tag{1}$$

The sequence in (1) can also be written as

$$2 + 6x + 12x^2 + 20x^3 + 30x^4 + \cdots + 72x^7 + 90x^8 + \cdots \tag{2}$$

which displays the generating series of W_n. The generating series is a powerful tool for encoding information about an integer sequence and effectively tracks each term in the sequence [30]. For instance, if we write the sequence as written in (1), it is impossible to determine which term 72 is. However, if we wrote the generating series instead, we would have the sequence as written in (2) which clearly shows that 72 is the seventh term of the sequence, that is, $W_7 = 72$.

Note that when discussing generating series, we always start our sequence with a_0. Since a sequence can be manipulated algebraically, the sequence in (1) can also be written as a generating function. When we have one generating function, we can get another generating function for a variety of sequences.

For example, we can write the generating series of a sequence 1, 1, 1, 1, 1, ... as $1 + x + x^2 + x^3 + x^4 + \cdots$. Using the differencing technique [30]. We have the generating function for the above generating series, which is shown below. The differencing technique consists of three main steps, i.e., "multiple-shift-subtract." The ideas are multiplying the generating series, G by variable x to obtain Gx; shift over the sum of Gx to the right, and get the subtraction between these two.

$$\begin{aligned} G = &\quad 1 + x + x^2 + x^3 + x^4 + \cdots \\ -Gx = &\quad x + x^2 + x^3 + x^4 + \cdots \\ (1-x)G = &\quad 1 \\ G = &\quad \frac{1}{1-x}. \end{aligned}$$

Hence, we have:

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \tag{3}$$

which shown that the generating function for the sequence 1, 1, 1, 1, 1, … is $\frac{1}{1-x}$. If we take the first derivative of $\frac{1}{1-x}$, we get $\frac{1}{(1-x)^2}$. Then, if we differentiate each term in the power series, we get $(1 + x + x^2 + x^3 + x^4 + x^5 \ldots)' = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \ldots$ which is the generating series for the sequence 1, 2, 3, 4, …, which gives us the following.

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \frac{1}{(1-x)^2} \tag{4}$$

Theorem 2. For any $n \geq 0$, the generating function of W_n can be expressed as $2 + 6x + 12x^2 + 20x^3 + 30x^4 + \cdots = \frac{2}{(1-x)^3}$.

Proof.

We denote the generating function as G. From (2), we have the generating series $G = 2x^0 + 6x^1 + 12x^2 + 20x^3 +$

$30x^4 + \cdots$. Using the multiply-shift-subtract technique [30], we have

$$\begin{aligned} G &= 2x^0 + 6x^1 + 12x^2 + 20x^3 + 30x^4 + \cdots \\ (-)\ xG &= 2x^1 + 6x^2 + 12x^3 + 20x^4 + \cdots \\ \hline (1-x)G &= 2x^0 + 4x^1 + 6x^2 + 8x^3 + 10x^4 + \cdots \\ (1-x)G &= 2 + 4x + 6x^2 + 8x^3 + 10x^4 + \cdots \\ (1-x)G &= 2 + 2(2x + 3x^2 + 4x^3 + 5x^4 + \cdots) \end{aligned}$$

From (4), we have

$$\begin{aligned} (1-x)G &= 2 + 2\left(\frac{1}{(1-x)^2} - 1\right) \\ (1-x)G &= 2 + \frac{2}{(1-x)^2} - 2 \\ (1-x)G &= \frac{2}{(1-x)^2} \\ G &= \frac{2}{(1-x)^2} \cdot \frac{1}{(1-x)} \\ G &= \frac{2}{(1-x)^3} \end{aligned}$$

Theorem 3. For any $n \geq 0$, the recurrence relation of W_n is given by $W_n = W_{n-1} + 2(n+1)$.

Proof.

From Theorem 2, we have:

$$\begin{aligned} A &= \frac{2}{(1-x)^3} \\ &= 2\left(\frac{2}{(1-x)^3}\right) \\ &= 2(1 + 3x + 6x^2 + 10x^3 + \cdots) \\ &= 2 + 6x + 12x^2 + 20x^3 + 30x^4 + 42x^5 + \cdots \end{aligned}$$

which provide the sequence $2, 6, 12, 20, 30, 42, \ldots$. Observing the pattern among these terms, we have:

$$\begin{aligned} W_0 &= 2 \\ W_1 &= 2 + 2(2) = 6 \\ W_2 &= 6 + 2(3) = 12 \\ W_3 &= 12 + 2(4) = 20 \\ W_4 &= 20 + 2(5) = 30 \\ &\vdots \qquad\quad \vdots \\ W_n &= W_{n-1} + 2(n+1) \end{aligned}$$

For reference, the first 40 terms of W_n is listed in Table 4.

TABLE IV
THE FIRST FORTY TERMS OF THE WING SEQUENCE

W_0	2	W_{10}	132	W_{20}	462	W_{30}	992
W_1	6	W_{11}	156	W_{21}	506	W_{31}	1056
W_2	12	W_{12}	182	W_{22}	552	W_{32}	1122
W_3	20	W_{13}	210	W_{23}	600	W_{33}	1190
W_4	30	W_{14}	240	W_{24}	650	W_{34}	1260
W_5	42	W_{15}	272	W_{25}	702	W_{35}	1332
W_6	56	W_{16}	306	W_{26}	756	W_{36}	1406
W_7	72	W_{17}	342	W_{27}	812	W_{37}	1482
W_8	90	W_{18}	380	W_{28}	870	W_{38}	1560
W_9	110	W_{19}	420	W_{29}	930	W_{39}	1640

IV. CONCLUSION

In conclusion, this study has successfully explored and extended the geometric representation of the Motzkin sequence to develop a new sequence, the Wing sequence. By modifying the geometric representation of non-intersecting chords on a circle, we have derived the Wing sequence, W_n, which is represented as $2, 6, 12, 20, 30$, and so on. Each term in the sequence corresponds to the number of unique directed paths that visit $n+3$ points at least once. For instance, $W_0 = 2$ signifies that there are 2 distinct ways to draw unique directed paths among 3 points, while $W_1 = 6$ indicates that there are 6 unique directed paths among 4 points. In general, there are $(n+1)(n+2)$ unique directed paths among $n+3$ points.

The primary contributions of this research are threefold. Firstly, we have derived a closed-form formula for W_n, expressed as $W_n = (n+1)(n+2)$ and established a recurrence relation $W_n = W_{n-1} + 2(n+1)$. Secondly, we have developed a generating function for the Wing sequence, $G(x) = \frac{2}{(1-x)^3}$, which provides a powerful tool for encoding and analyzing each term of a sequence. Thirdly, our study has introduced and validated the FSTP (First-Second-Third-Point) technique, a novel method for constructing distinct directed paths.

In addition to the aforementioned findings, researchers should continue to investigate the properties of the sequence, particularly the specific characteristics of each term in the Wing sequence. Furthermore, these findings could be investigated for their potential applications in combinatorial structures such as network routing, project scheduling, and key generation for data security.

Geometrically, the Motzkin sequence pertains to counting non-intersecting chords that connect points on a circle, whereas the Wing sequence counts directed paths that visit all points in linear arrangement exactly once. Although both sequences are related to geometric representations to develop a sequence, our observation shows that the Motzkin sequence provides a more straightforward framework for understanding geometric representation, making it easier for researchers to create new ideas. Meanwhile, the Wing sequence offers a systematic technique that identifies patterns among the first, second, and third points to derive the sequence.

In summary, this study introduces a theoretical tool to develop a sequence using the idea of geometric representation from the non-intersecting chords on a circle. The tool has potential applications and highlights opportunities for further research.

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