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Investigation of the hyperchaos and control in the fractional order financial system with profit margin

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ABSTRACT

This research study proposes a novel hyperchaotic finance system with profit margin and then utilizes the Adomian Decomposition Method (ADM) to tackle the solution of the associated Caputo-derivative based fractional-order hyperchaotic finance system with profit margin. Numerical simulations and analyses, such as the evaluation of Lyapunov exponents (LE), the creation of bifurcation diagrams (BD), the complexity analysis (CA), and the 0-1 test, are used to get a full picture of the system. The results of our study demonstrate the occurrence of periodic, chaotic, and hyperchaotic dynamics in the system. Furthermore, the overall criteria for the existence and the uniqueness of the exact solutions for Caputo fractional-order models are presented. In addition, we present a control methodology for the fractional-order hyperchaotic financial system utilizing the Laplace transform and linear feedback control. Significantly, our research showcases a noteworthy correlation between the analytical results and numerical simulations, emphasizing the soundness and effectiveness of our methodology.

1. Introduction

Fractional calculus is an area in mathematics generalizing the concepts of integration and differentiation to non-integer order.[1] In traditional calculus, differentiation corresponds to finding the rate of change of functions, and integration corresponds to finding the accumulation of a quantity over an interval.[2] In fractional calculus, these operations are extended to allow for differentiation and integration with non-integer orders, typically expressed as a fractional power of the derivative or integral operator.[3] Fractional calculus has applications in various fields, including finance,[4] biology,[5] robotic,[6] encryption,[7] physics[8] and electronic circuit.[9]

The fractional calculus offers a more flexible and versatile framework that can provide additional insights and better describe economic of behaviors found in chaotic systems.[10–12] Chaotic systems often exhibit long-range dependencies and memory effects, where the behavior of the system at one time affects its behavior at much later times.[13] Fractional calculus allows for the modeling of such behaviors more naturally by incorporating non-integer orders of differentiation and integration.[14]

In the financial context, hyperchaotic systems can arise when modeling complex interactions between various financial variables, such as asset prices, interest rates, and market indicators.[15] These systems are of interest because they can capture extreme fluctuations, abrupt

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changes, and irregular patterns that are sometimes observed in financial markets.[16] Financial systems are influenced by a multitude of factors, including economic indicators, investor sentiment, policy changes, and external events.[17] Hyperchaotic models attempt to capture the intricate interactions between these factors, leading to complex and unpredictable dynamics.[18]

Hyperchaotic behavior in financial systems can have real-world consequences for investors, traders, regulators, and policymakers. Yu et al.[19] presented some novel hyperchaotic systems within a financial model involving profit margins. They explored the effectiveness of linear and speed feedback controllers that stabilizes hyperchaos toward unstable equilibrium points. Vargas et al.[20] presented new adaptive procedure to achieve synchronization in some hyperchaotic financial systems, accounting for unknown system parameters and bounded disturbances. Szumiński[21] formulated two hyperchaotic financial systems and provided analytical evidence for these assertions by evaluating conditions of variational equations of differential Galois groups along specific solutions. Kai et al.[22] introduced a financial hyperchaotic system of four-dimensional derived from an established three-dimensional nonlinear financial model, enhanced with controller term that can address the impact of control on the systems. Their study revealed the occurrence of a positively invariant set known as Hopf bifurcation, and a means of implementation using electronic circuits. Rao and Zhu[23] simultaneously addressed stabilization and synchronization for hyperchaotic financial systems with double delays. A novel global Mittag-Leffler synchronization condition for financial hyperchaotic system of fractional-order was developed by Li et al.[24] This scheme is achieved through the design of appropriate impulsive control and state feedback controllers. Yousri and Mirjalili[25] proposed a robust optimization algorithm for identifying parameters governing fractional-order chaotic behavior, noise-influenced hyperchaotic and chaotic financial system. Xu et al.[26] tackled the hyperchaotic system's $H\infty$ control problems subject to energy-bounded disturbances based on delayed feedback controllers. By considering the homogeneous Neumann boundary conditions on a set of semi-linear parabolic partial differential equations, Wang et al.[27] investigated the financial system that included sub-blocks of the labour force, stocks, money, and production distributed along particular line segments or planar areas.

In the recent decades, there is significant interest in the financial literature in the modeling of Caputo derivative based fractional finance systems. Olayiwola et al.[28] proposed a Caputo fractional order financial mathematical model and investigated the impact of minimum interest rate and maximum investment demand in the proposed financial model. Aljethi and Kilicman[29] proposed a fractional differential equation and studied its application to realistic finance data. Alzaid et al.[30] studied the chaotic behavior of a new fractional financial mathematical model developed with a generalized Caputo derivative operator. Li et al.[31] studied a Caputo derivative based fractional financial system modeling the dynamics of a three-agent financial bubble. In the recent years, research works have also paid attention to Caputo derivative based fractional hyperchaotic finance systems. Sene[32] proposed a new fractional 4-D hyperchaotic finance system using Caputo-Liouville derivative and investigated the existence and uniqueness of solutions of the proposed 4-D fractional hyperchaotic finance model. Hajipour et al.[33] developed a fractional hyperchaotic scheme characterized by disturbances and uncertain dynamics, demonstrating the efficacy of the robust adaptive sliding mode control system.

Motivated by the above literature, this study aims to address the solution of fractional-order financial systems incorporating profit margin using ADM strategy.

The main contribution and novelty of this work as follows:

1. System hyperchaotic financial system has 11 terms with three quadratic nonlinearities.
2. Solving fractional-order financial system incorporating profit margin using Adomian decomposition method (ADM).
3. System exhibits periodic, chaotic, and hyperchaotic behavior
4. The control strategy for the fractional-order hyperchaotic financial systems using Laplace transform and linear feedback control was studied.

This work is systematized as follows. The next Section will present the preliminaries, algorithms, and model derivation. In Section 4, we investigate the dynamical behavior in fractional-order hyperchaotic financial scheme. Furthermore, in Section 4, we propose a control scheme for hyperchaotic financial systems of fractional-order, employing Laplace transform and linear feedback control. The study is summarized and a concluding remark is presented in the last section.

2. Motivation and reasons for using Caputo based model

Traditional integer-order calculus is generalized to non-integer orders in fractional calculus. Several recent studies, for example,[34–43] can be consulted to understand the importance and uses of fractional calculus in many fields of science and engineering. The Caputo fractional operator is often used as a fractional derivative in mathematical modeling. Several causes have prompted the adoption of the Caputo fractional operator in modeling, and early results suggest that it can more precisely capture the behavior of a wide range of systems than standard integer-order models. Here is a comprehensive analysis of the rationale behind using the Caputo fractional operators in mathematical modeling:

- **Historical Context:** The history of fractional calculus stretches back centuries, at least to the time of Leibniz. However, until it was understood that some phenomena could not be correctly represented by integer-order differential equations, they remained primarily in the realm of mathematical curiosity.
- **Replacement with Fractional-order:** The famous Riemann–Liouville integral formula can be shown to be developed from the Cauchy method for repeated integration by merely substituting $r \in \mathbb{C}$, $Re(r) > 0$ for $n \in \mathbb{N}$ and it is this fractional integral formula that has served as the basic for the developing different types of numerical methods for solving numerous partial and fractional ordinary differential equations.
- **Effects on memory and heredity:** Memory is an inherent property of fractional-order systems. Systems with memory or hereditary effects can be modeled with the help of the Caputo fractional operator. Such systems remember their prior states and behaviors and use that data to inform their current behavior. Normal differential equations of integer order are inadequate to capture this phenomenon.
- **Kinematical Complexity:** Systems with complex dynamics can be described using fractional differential equations in their simpler versions. System features like power laws, long-range dependencies, and anomalous diffusion are often better captured by fractional-order models.
- **Meaning in the Real World:** The use of fractional derivatives can be understood physically in particular contexts. In the case of anomalous diffusion, for instance, one can directly relate the diffusion's nature (sub-diffusion vs. super-diffusion) to the exponent of the fractional derivative.
- **Modeling Adaptability:** The order of the fractional derivative makes an additional modeling degree of freedom available. This expands the possible system behaviors, letting us fine-tune the model to better reflect real-world data.
- **Improved Agreement with Experimental Data:** Many real-world studies have shown that models that use the Caputo fractional operator can fit experimental data better than integer-order models, especially when the data has non-standard dynamics.
- **Nonlocal Nature:** Nonlocality is an intrinsic property of fractional operators, which means that the previous history of the system has an effect on the current state of the system. Some systems' behaviors may be difficult to capture without this feature.

- **Intermediate Processes:** Some processes cannot be described using the usual differential or integral notation. The Caputo operator provides a transitional state between these two extremes of differentiation and integration.
- **Classical Models Generalized:** In many cases, the classical integer-order model can be retrieved as a special case by switching to fractional calculus and therefore generalizing the theory.
- **Mathematical Consistency:** Among fractional derivatives, the Caputo derivative is notably helpful because it offers causality in the time domain, guaranteeing that the current state of a system depends solely on the past and not the future. For many physical and engineering uses, this is essential.

It is important to note that the benefits and enhanced precision of modeling specific processes do not come without the obstacles posed by fractional calculus. Fractional-order differential equations can be more difficult to solve analytically and numerically than their integer-order equivalents. In addition, the fractional order is not always easy to comprehend physically. The increasing focus on this topic, however, suggests that it may help us better comprehend other complex systems.

- By considering the literature, it is obvious to notice that the well-known Riemann–Liouville integral scheme is derived from the Cauchy procedure for the repeated integration where Riemann–Liouville notably replaced $n \in \mathbb{N}$ with $r \in \mathbb{C}, Re(r) > 0$ and recently, the fractional integral method has been the cornerstone for developing of different kinds of numerical schemes for obtaining the solution of partial and fractional ordinary differential equations.

After being motivated with continuous advancements of fractional calculus in various fields of applied sciences particularly use of Caputo differential operator, we present the Caputo variant of Eq. (2.1) in the form given below whereas the justification and motivation for using this fractional operator is also detailed in the forthcoming section.

$$\begin{cases} {}^{\mathrm{C}}\mathbb{D}^r_{t_0,t}x(t) = \dfrac{1}{\Gamma(1-r)}\int_0^t (t-\zeta)^{-r}x'(\zeta)d\zeta = (y-a)x + z + qw, \\ {}^{\mathrm{C}}\mathbb{D}^r_{t_0,t}y(t) = \dfrac{1}{\Gamma(1-r)}\int_0^t (t-\zeta)^{-r}y'(\zeta)d\zeta = 1 - by - x^2, \\ {}^{\mathrm{C}}\mathbb{D}^r_{t_0,t}z(t) = \dfrac{1}{\Gamma(1-r)}\int_0^t (t-\zeta)^{-r}z'(\zeta)d\zeta = -x - cz, \\ {}^{\mathrm{C}}\mathbb{D}^r_{t_0,t}w(t) = \dfrac{1}{\Gamma(1-r)}\int_0^t (t-\zeta)^{-r}w'(\zeta)d\zeta = -dxy - pz, \end{cases} \tag{2.1}$$

with the initial conditions as defined below:

$$x(0) \geq 0,\ y(0) \geq 0,\ z(0) \geq 0,\ w(0) \geq 0. \tag{2.2}$$

3. Preliminaries, algorithms and modeling

3.1. Adomian decomposition method

For the fractional-order continuous system, a number of numerical solution schemes are available. However, some of the classical and widely used algorithms are the Adomian decomposition method (ADM), Adams–Bashforth–Moulton scheme (ABM), and the frequency domain method (FDM).[44,45] The ADM is used in this paper since it has higher computational accuracy and its algorithm is presented as follows.

Consider the nonlinear differential equation of fractional-order

$${}^{C}_{t_0}D^r_t x(t) = f(x(t)) + g(t), \tag{3.1}$$

with ${}^{C}_{t_0}D^r_t$ representing the Caputo derivative operator Eq. (3.1), $\mathbf{g}(t) = [g_1(t), g_2(t), \dots, g_n(t)]^T$ denoting the constants, and $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ defining the state variables. Eq. (3.1) can be decomposed as follows:

$${}^{C}_{t_0}D^r_t x(t) = Lx(t) + Nx(t) + g(t), \tag{3.2}$$

with $Nx(t)$ denoting the nonlinear term of $fx(t)$, while $L_x(t)$ is the linear terms of $f(x(t))$. ${}_{t_0}J^r_t$ is the integral operator, and applying it in Eq. (3.3) will produce:

$$x(t) = {}_{t_0}J^r_t Lx(t) + {}_{t_0}J^r_t Nx(t) + {}_{t_0}J^r_t g(t) + \sum_{k=0}^{m-1} b_k \frac{(t-t_0)^k}{k!}, \tag{3.3}$$

with the following initial condition $k = 0, \dots, m-1$, and $\mathbf{x}^{(k)}\left(t_0^+\right) = \mathbf{b}_k$. Here, $Nx(t)$ can be evaluated by

$$Nx(t) = \sum_{i=0}^{\infty} A^i(x^0, x^1, \dots, x^i), \tag{3.4}$$

where

$$\begin{cases} A^i_j = \frac{1}{i!}\left[\frac{d^i}{d\lambda^i}N(v^i_j(\lambda))\right]_{\lambda=0} \\ v^i_j(\lambda) = \sum_{k=0}^{i}(\lambda)^k\ x^k_j, \end{cases} \tag{3.5}$$

for $i = 0, 1, 2, \dots, \infty$, with $j = 1, \dots, n$ and

$$\begin{cases} x^0 = {}_{t_0}J^r_t g \sum_{k=0}^{m-1} b_k\ \frac{(t-t_0)^k}{k!}, \\ x^1 = {}_{t_0}J^r_t Lx^0 + {}_{t_0}J^r_t A^0(x^0), \\ x^2 = {}_{t_0}J^r_t Lx^1 + {}_{t_0}J^r_t A^1(x^0, x^1), \\ \dots, \dots, \dots, \\ x^i = {}_{t_0}J^r_t Lx^{i-1} + {}_{t_0}J^r_t A^{i-1}(x^0, x^1, \dots, x^{i-1}) \\ \dots, \dots, \dots. \end{cases} \tag{3.6}$$

Therefore, the numerical solution is obtained:

$$\mathbf{x}(t) = \sum_{i=0}^{\infty} x^i. \tag{3.7}$$

3.2. Complexity analysis

An important method for examining the dynamic of chaotic systems is the complexity measure of nonlinear time series. The subject is presently an interesting and widely studied topic in the area of nonlinear system. This method employs a number of algorithms to determine the closeness some chaotic sequences to other random sequences, with the complexity value increasing as the sequence gets nearer to the random sequence. Some of the widely considered algorithms for these problems includes C_0 algorithm and spectral entropy (SE), sample entropy, fuzzy entropy, and statistical complexity measure (SCM). However, C_0 and SE schemes can precisely assess time series complexity and thus, in this study, we will consider both C_0 algorithm and spectral entropy (SE).

Consider the following time series, $\{x(n), n = 0, 1, 2, \dots, N-1\}$. By eliminating the average value, we obtain another time series of the form:

$$x(n) = x(n) - \overline{x}, \tag{3.8}$$

with $\overline{x} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$. Eq. (3.9) has the following Fourier transform:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk}, \tag{3.9}$$

for $k = 0, 1, 2, \dots, N-1$.

By setting

$$P(k) = \frac{|X(k)|^2}{\sum_{k=0}^{\frac{N}{2}-1}|X(k)|^2}, \tag{3.10}$$

we define the following spectral entropy (SE):

$$SE = -\frac{1}{\ln\left(\frac{N}{2}\right)} \cdot \sum_{k=0}^{\frac{N}{2}-1} P(k) \cdot \ln(P(k)). \tag{3.11}$$

We set

$$G_N = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2, \tag{3.12}$$

We introduce a control parameter r. Thus, we can generalize the new series into

$$\overline{X}(k) = \begin{cases} X(k), if \mid X(k) \mid^2 > rG_N, \\ 0, if \mid X(k) \mid^2 \leq rG_N, \end{cases} \tag{3.13}$$

Then, the C_0 complexity can be defined as

$$C_0(r, N) = \frac{\sum_{n=0}^{N-1} |x(n) - \overline{x}(n)|^2}{\sum_{n=0}^{N-1} |x(n)|^2}, \tag{3.14}$$

with $n = 0, 1, \ldots, N-1$ and $\overline{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \overline{X}(k) e^{\frac{j2\pi nk}{N}}$

An efficient and reliable binary testing approach to find chaos in different systems is the 0–1 test. The fundamental concept is to build a data stochastic dynamic scheme and then observe how magnitude of that scheme varies over time. While unbounded paths, such as Brownian motion, demonstrate the chaotic state of the dynamic scheme, the p-s plane bounded paths suggest the dynamic scheme is a regular system.

Consider the time series, $\{x(n), n = 0, 1, 2, \ldots, N-1\}$ with the following defined coordinate points $\left(p_c(n), s_c(n)\right)$:

$$\begin{cases} p_c(n) = \sum_{j=1}^{n} x(j) cos(\theta(j)), \\ s_c(n) = \sum_{j=1}^{n} x(j) sin(\theta(j)), \end{cases} \tag{3.15}$$

with $\theta(j) = jc + \sum_{i=1}^{j} x(i), c \in \left[\frac{\pi}{5}, \frac{4\pi}{5}\right]$.

3.3. Modeling

In 2001, Ma and Chen[46] proposed a third-order dynamic model that characterizes a nonlinear chaotic financial system. This model delineates the temporal changes in three state variables: the interest rate (x), investment demand (y), and the price index (z). The set of differential equations governing the Ma-Chen finance system[46] is expressed as follows:

$$\begin{cases} \dot{x} &= z + (y - a)x \\ \dot{y} &= 1 - by - x^2 \\ \dot{z} &= -x - cz \end{cases} \tag{3.16}$$

In (3.16), a represents savings, b denotes the per-investment cost, and c signifies the elasticity of the demands of commercials. These parameters are characterized as positive constants.

The proposition of a novel hyperchaotic finance system, distinguished by two positive Lyapunov exponents, includes the integration of an extra state variable u, symbolizing the average profit margin as proposed by Yu et al.[19] The Yu hyperchaotic finance system[19] is delineated by a set of autonomous differential equations given as follows:

$$\begin{cases} \dot{x} &= z + (y - a)x + u \\ \dot{y} &= 1 - by - x^2 \\ \dot{z} &= -x - cz \\ \dot{u} &= -dxy - ku \end{cases} \tag{3.17}$$

By modifying the dynamics of the Yu hyperchaotic finance system (3.17), we obtain a new hyperchaotic finance system in this research work.

Next, we present a novel 4D financial hyperchaotic system with profit margin, which is delineated by a set of autonomous differential equations given as follows:

$$\begin{cases} \dot{x} &= (y - a)x + z + qw \\ \dot{y} &= 1 - by - x^2 \\ \dot{z} &= -x - cz \\ \dot{w} &= -dxy - pz \end{cases} \tag{3.18}$$

for some positive constants a, b, c, d, p, q.

Suppose the chosen values of a, b, c, d, p, q are:

$$a = 1, b = 0.1, c = 0.8, d = 0.4, p = 0.3, q = 0.2 \tag{3.19}$$

By considering the initial points of $x(0) = 0.6$, $y(0) = 0.4$, $z(0) = 0.6$ and $w(0) = 0.4$, we obtain the Lyapunov exponents (LE) of system (3.18) as follows $\tau_1 = 0.0775$, $\tau_2 = 0.0237$, $\tau_3 = 0$ and $\tau_4 = -0.3512$ to illustrate the hyperchaotic behavior. Fig. 1 exhibits the plot for phase portraits and strange attractors of the system. Let us consider the following fractional-order system:

$$\begin{cases} {}_{t_0}^{C}D_t^r x = (y - a)x + z + qw, \\ {}_{t_0}^{C}D_t^r y = 1 - by - x^2, \\ {}_{t_0}^{C}D_t^r z = -x - cz, \\ {}_{t_0}^{C}D_t^r w = -dxy - pz, \end{cases} \tag{3.20}$$

with the order of derivative given as $r \in (0, 1)$. Obviously, with $r = 1$ system (3.20) reduces to model (3.18).

Based on ADM scheme, system (3.20) becomes:

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} x(t_0) \\ y(t_0) \\ z(t_0) \\ w(t_0) \end{bmatrix} + {}_{t_0}I_t^r \begin{bmatrix} -ax + z + qw \\ 1 - by \\ -x - cz \\ -pz \end{bmatrix} + {}_{t_0}I_t^r \begin{bmatrix} xy \\ -x^2 \\ 0 \\ -dxy \end{bmatrix} \tag{3.21}$$

with the decomposed nonlinear terms given as:

$$\begin{cases} A^0_{-x^2} = -x^0x^0 \\ A^1_{-x^2} = -(x^1x^0 + x^0x^1) \\ A^2_{-x^2} = -(x^2x^0 + x^1x^1 + x^0x^2) \\ A^3_{-x^2} = -(x^3x^0 + x^2x^1 + x^1x^2 + x^0x^3) \\ A^4_{-x^2} = -(x^4x^0 + x^3x^1 + x^2x^2 + x^1x^3 + x^0x^4) \\ A^5_{-x^2} = -(x^5x^0 + x^4x^1 + x^3x^2 + x^2x^3 + x^1x^4 + x^0x^5) \end{cases} \tag{3.22}$$

$$\begin{cases} A^0_{xy} = x^0y^0 \\ A^1_{xy} = x^1y^0 + x^0y^1 \\ A^2_{xy} = x^2y^0 + x^1y^1 + x^0y^2 \\ A^3_{xy} = x^3y^0 + x^2y^1 + x^1y^2 + x^0y^3 \\ A^4_{xy} = x^4y^0 + x^3y^1 + x^2y^2 + x^1y^3 + x^0y^4 \\ A^5_{xy} = x^5y^0 + x^4y^1 + x^3y^2 + x^2y^3 + x^1y^4 + x^0y^5 \end{cases} \tag{3.23}$$

where the number of ADM decomposition is denoted by the superscript in the decomposition system. For these systems, we have the following initial states:

$$\begin{cases} c_1^0 = x(t_0) \\ c_2^0 = y(t_0) \\ c_3^0 = z(t_0) \\ c_4^0 = w(t_0) \end{cases} \tag{3.24}$$

with the coefficients set given as

$$\begin{cases} c_1^1 = qc_4^0 - ac_1^0 + c_3^0 + c_1^0c_2^0 \\ c_2^1 = 1 - bc_2^0 - c_1^0c_1^0 \\ c_3^1 = -c_1^0 - cc_3^0 \\ c_4^1 = -pc_3^0 - dc_1^0c_2^0 \end{cases} \tag{3.25}$$

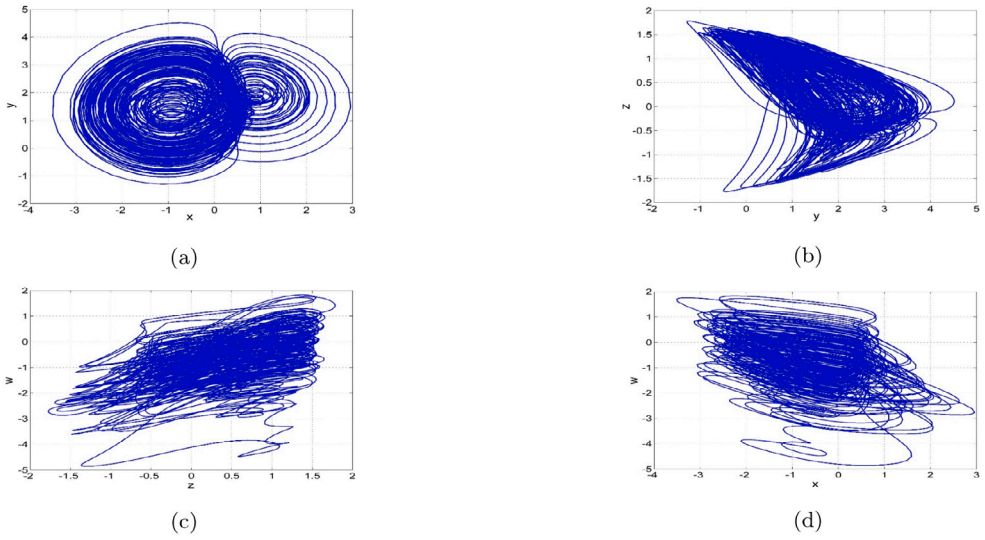


Fig. 1. Hyperchaotic financial system (3.18) using MATLAB in (a) x-y plane (b) y-z plane, (c) z-w plane and (d) x-w plane.

$$\begin{cases} c_1^2 = qc_4^1 - ac_1^1 + c_3^1 + c_1^1c_2^0 + c_1^0c_2^1 \\ c_2^2 = 1 - bc_2^1 - c_1^1c_1^0 - c_1^0c_1^1 \\ c_3^2 = -c_1^1 - cc_3^1 \\ c_4^2 = -pc_3^1 - d\left(c_1^1c_2^0 + c_1^0c_2^1\right) \end{cases} \tag{3.26}$$

$$\begin{cases} c_1^3 = qc_4^2 - ac_1^2 + c_3^2 + c_1^0c_2^2 + c_1^2c_2^0 + c_1^1c_2^1\frac{\Gamma(2r+1)}{\Gamma^2(r+1)} \\ c_2^3 = 1 - bc_2^2 - c_1^0c_1^2 - c_1^2c_1^0 - c_1^1c_1^1\frac{\Gamma(2r+1)}{\Gamma^2(r+1)} \\ c_3^3 = -c_1^2 - cc_3^2 \\ c_4^3 = -pc_3^2 - d\left((c_1^0c_2^2 + c_1^2c_2^0 + c_1^1c_2^1)\frac{\Gamma(2r+1)}{\Gamma^2(r+1)}\right) \end{cases} \tag{3.27}$$

$$\begin{cases} c_1^4 = qc_4^3 - ac_1^3 + c_3^3 + c_1^3c_2^0 + c_1^0c_2^3 + (c_1^1c_2^2 + c_1^2c_2^1)\frac{\Gamma(3r+1)}{\Gamma(r+1)\Gamma(2r+1)} \\ c_2^4 = 1 - bc_2^3 - c_1^3c_1^0 - c_1^0c_1^3 - (c_1^1c_1^2 + c_1^2c_1^1)\frac{\Gamma(3r+1)}{\Gamma(r+1)\Gamma(2r+1)} \\ c_3^4 = -c_1^3 - cc_3^3 \\ c_4^4 = -pc_3^3 - d\left(c_1^3c_2^0 + c_1^0c_2^3 + (c_1^1c_2^2 + c_1^2c_2^1)\frac{\Gamma(3r+1)}{\Gamma(r+1)\Gamma(2r+1)}\right) \end{cases} \tag{3.28}$$

$$\begin{cases} c_1^5 = qc_4^4 - ac_1^4 + c_3^4 + c_1^4c_2^0 + c_1^0c_2^4 + (c_1^3c_2^1 + c_1^1c_2^3)\frac{\Gamma(4r+1)}{\Gamma(r+1)\Gamma(3r+1)} + c_1^2c_2^2\frac{\Gamma(4r+1)}{\Gamma^2(2r+1)} \\ c_2^5 = 1 - bc_2^4 - c_1^4c_1^0 - c_1^0c_1^4 - (c_1^3c_1^1 + c_1^1c_1^3)\frac{\Gamma(4r+1)}{\Gamma(r+1)\Gamma(3r+1)} - c_1^2c_1^2\frac{\Gamma(4r+1)}{\Gamma^2(2r+1)} \\ c_3^5 = -c_1^4 - cc_3^4 \\ c_4^5 = -pc_3^4 - d\left(c_1^4c_2^0 + c_1^0c_2^4 + (c_1^3c_2^1 + c_1^1c_2^3)\frac{\Gamma(4r+1)}{\Gamma(r+1)\Gamma(3r+1)} + c_1^2c_2^2\frac{\Gamma(4r+1)}{\Gamma^2(2r+1)}\right) \end{cases} \tag{3.29}$$

$$\begin{cases} c_1^6 &= qc_4^5 - ac_1^5 + c_3^5 + c_1^5c_2^0 + c_1^0c_2^5 + (c_1^4c_2^1 + c_1^1c_2^4)\frac{\Gamma(5r+1)}{\Gamma(r+1)\Gamma(4r+1)} \\ &\quad + (c_1^3c_2^2 + c_1^2c_2^3)\frac{\Gamma(5r+1)}{\Gamma(2r+1)\Gamma(3r+1)} \\ c_2^6 &= 1 - bc_2^5 - c_1^5c_1^0 - c_1^0c_1^5 - (c_1^4c_1^1 + c_1^1c_1^4)\frac{\Gamma(5r+1)}{\Gamma(r+1)\Gamma(4r+1)} \\ &\quad - (c_1^3c_1^2 + c_1^2c_1^3)\frac{\Gamma(5r+1)}{\Gamma(2r+1)\Gamma(3r+1)} \\ c_3^6 &= -c_1^5 - cc_3^5 \\ c_4^6 &= -pc_3^5 - d\left(c_1^5c_2^0 + c_1^0c_2^5 + (c_1^4c_2^1 + c_1^1c_2^4)\frac{\Gamma(5r+1)}{\Gamma(r+1)\Gamma(4r+1)}\right. \\ &\quad \left. + (c_1^3c_2^2 + c_1^2c_2^3)\frac{\Gamma(5r+1)}{\Gamma(2r+1)\Gamma(3r+1)}\right) \end{cases} \tag{3.30}$$

The numerical solution can be computed using the property of fractional-order calculus as follows:

$$\begin{cases} x(t) &= c_1^0 + c_1^1\frac{h^r}{\Gamma(r+1)} + c_1^2\frac{h^{2r}}{\Gamma(2r+1)} + c_1^3\frac{h^{3r}}{\Gamma(3r+1)} \\ &\quad + c_1^4\frac{h^{4r}}{\Gamma(4r+1)} + c_1^5\frac{h^{5r}}{\Gamma(5r+1)} + c_1^6\frac{h^{6r}}{\Gamma(6r+1)} \\ y(t) &= c_2^0 + c_2^1\frac{h^r}{\Gamma(r+1)} + c_2^2\frac{h^{2r}}{\Gamma(2r+1)} + c_2^3\frac{h^{3r}}{\Gamma(3r+1)} \\ &\quad + c_2^4\frac{h^{4r}}{\Gamma(4r+1)} + c_2^5\frac{h^{5r}}{\Gamma(5r+1)} + c_2^6\frac{h^{6r}}{\Gamma(6r+1)} \\ z(t) &= c_3^0 + c_3^1\frac{h^r}{\Gamma(r+1)} + c_3^2\frac{h^{2r}}{\Gamma(2r+1)} + c_3^3\frac{h^{3r}}{\Gamma(3r+1)} \\ &\quad + c_3^4\frac{h^{4r}}{\Gamma(4r+1)} + c_3^5\frac{h^{5r}}{\Gamma(5r+1)} + c_3^6\frac{h^{6r}}{\Gamma(6r+1)} \\ w(t) &= c_4^0 + c_4^1\frac{h^r}{\Gamma(r+1)} + c_4^2\frac{h^{2r}}{\Gamma(2r+1)} + c_4^3\frac{h^{3r}}{\Gamma(3r+1)} \\ &\quad + c_4^4\frac{h^{4r}}{\Gamma(4r+1)} + c_4^5\frac{h^{5r}}{\Gamma(5r+1)} + c_4^6\frac{h^{6r}}{\Gamma(6r+1)} \end{cases} \tag{3.31}$$

with the step size denoted by h.

4. Analysis of dynamical characteristics

4.1. Existence and uniqueness of solutions

Fractional differential equations involve Caputo fractional derivatives, which are extensions of classical derivatives to non-integer orders. Due to the non-local character of fractional derivatives, the analysis of the existence and uniqueness of solutions for Caputo fractional-order models can be more complex than for integer-order ODEs. Several methods, including fixed-point theorems, integral equation methods, and tools from functional analysis, can be used to prove the existence of solutions for the Caputo fractional differential equation. There has been extensive study into proving the existence of results for fractional-order differential equations, which are useful for modeling physical phenomena involving memory effects and anomalous diffusion. Due to the non-locality of the fractional derivatives, showing the uniqueness of solutions for fractional-order differential equations might be more difficult than proving their existence. The uniqueness of the solution may be conditional on the original data's regularity and the constraints of the fractional differential equation.

Eq. (2.1) represents the novel form of the model in a fractional context using the nonlocal Caputo derivative. In this section, fixed-point theorems are employed to establish the existence and uniqueness of the proposed model's solution. To enhance the academic rigor of our proposed model, it is suggested that the model be reformulated with the

following structure:

$$\begin{cases} {}^{C}\mathbb{D}^{r}_{0,t}x(t) = \zeta_1(t,x,y,z,w), \\ {}^{C}\mathbb{D}^{r}_{0,t}y(t) = \zeta_2(t,x,y,z,w), \\ {}^{C}\mathbb{D}^{r}_{0,t}z(t) = \zeta_3(t,x,y,z,w), \\ {}^{C}\mathbb{D}^{r}_{0,t}w(t) = \zeta_4(t,x,y,z,w), \end{cases} \tag{4.1}$$

where the mathematical notations used on the right side of the above structure are set as follows:

$$\begin{cases} \zeta_1(t,x,y,z,w) = (y-a)x + z + qw, \\ \zeta_2(t,x,y,z,w) = 1 - by - x^2, \\ \zeta_3(t,x,y,z,w) = -x - cz, \\ \zeta_4(t,x,y,z,w) = -dxy - pz. \end{cases} \tag{4.2}$$

After setting up the notions, the Caputo model in (2.1) can also be rewritten as follows:

$$\begin{cases} {}^{C}\mathcal{D}^{r}_{0}\rho(t) = S(t,\rho(t)); \quad t \in J = [0,\beta], \quad 0 < r \leq 1, \\ \rho(0) = \rho_0 \geq 0. \end{cases} \tag{4.3}$$

The above setting is said to be valid only if the following structure holds true:

$$\begin{cases} \rho(t) = (x,y,z,w)^T, \\ \rho(0) = (x_0,y_0,z_0,w_0)^T, \\ W(t,\rho(t)) = (\zeta_i(t,x,y,z,w))^T, \quad i = 1,\ldots,4, \end{cases} \tag{4.4}$$

where the notation $(\cdot)^T$ used in the above setting stands for the transpose operation. After this, the Eq. (4.3) gets the following form:

$$\rho(t) = \rho_0 + \mathcal{J}^{r}_{0+}S(t,\rho(t)) = \rho_0 + \frac{1}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}S(\mu,\rho(\mu))d\mu. \tag{4.5}$$

Let us suppose that $\mathbb{E} = C([0,\beta];\mathbb{R})$ is a Banach space for all continuous functions from $\mathbb{R} \to [0,\beta]$, where $\|\rho\| = \sup_{t\in[0,\beta]}|\rho(t)|$. At this stage, we establish the following Theorem 1 to prove the uniqueness of the solution of the model based on the Caputo operator as shown in Eq. (2.1).

Theorem 1. *Let the function $S \in C([J,\mathbb{R}])$ and it maps bounded subset of $J \times \mathbb{R}^4$ into compact subset of $\mathbb{R}$. Further, it is also assumed that there is a fixed constant $\Xi_S > 0$ in such a way that the following condition (A_1) holds:*

(A_1) $\|S(t,\rho_1(t)) - S(t,\rho_2(t))\| \leq \Xi_S\|\rho_1(t) - \rho_2(t)\|$; $\forall\ t \in J$ and all $\rho_1,\rho_2 \in C([J,\mathbb{R}])$. Thus, whenever $\Theta\Xi_S < 1$, and $\Theta = \frac{\beta^r}{\Gamma(r+1)}$, then, the above Eq. (4.5) which is equivalent to Eq. (2.1) will produce a unique solution.

Proof. Let us consider $P : \mathbb{E} \to \mathbb{E}$ that is defined by the following equation:

$$(P\rho)(t) = \rho_0 + \frac{1}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}S(\mu,\rho(\mu))d\mu. \tag{4.6}$$

Now, P is well-defined and also a unique solution of Eq. (2.1). It further illustrate the fixed points of P. Obviously, by considering $\sup_{t\in J}\|S(t,0)\| = M_1$ and $\lambda \geq \|\rho_0\| + \Theta M_1$. Now, it suffices to justify $P\Pi_\lambda \subset \Pi_\lambda$, and $\Pi_\lambda = \{\rho \in \mathbb{E} : \|\rho\| \leq \lambda\}$, is convex and closed. Thus, each $\rho \in \Pi_\lambda$, we have

$$\begin{aligned} |(P\rho)(t)| &\leq |\rho_0| + \frac{1}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}|S(\mu,\rho(\mu))|d\mu \\ &\leq \rho_0 + \frac{1}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}\left[|S(\mu,\rho(\mu)) - S(\mu,0)| + |S(\mu,0)|\right]d\mu \\ &\leq \rho_0 + \frac{(\Xi_S\lambda + M_1)}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}d\mu \\ &\leq \rho_0 + \frac{(\Xi_S\lambda + M_1)}{\Gamma(r+1)}\beta^r \\ &\leq \rho_0 + \Theta(\Xi_S\lambda + M_1) \\ &\leq \lambda. \end{aligned} \tag{4.7}$$

We justify the results. Moreover, for $\rho_1,\rho_2 \in \mathbb{E}$, one attains

$$\begin{aligned} \|(P\rho_1)(t) - (P\rho_2)(t)\| &\leq \frac{1}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}\|S(\mu,\rho_1(\mu)) - S(\mu,\rho_2(\mu))\|d\mu \\ &\leq \frac{\Xi_S}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}\|\rho_1(\mu) - \rho_2(\mu)\|d\mu \\ &\leq \Theta\Xi_S\|\rho_1(t) - \rho_2(t)\|, \end{aligned} \tag{4.8}$$

this justifies that $\|(P\rho_1) - (P\rho_2)\| \leq \Theta\Xi_S\|\rho_1 - \rho_2\|$. Thus, due to Banach contraction, the unique solution for Eq. (2.1) is claimed. □

Now, the existence of the solutions of the Caputo model given in (2.1) will be shown with the aid of Krasnoselskii's fixed point justification as has been done in several existing studies, see for example.[47]

Lemma 1. *Let $\mathbb{E}$ be a Banach Space and $M \neq \emptyset$ defines a convex, closed, and bounded subset of $\mathbb{E}$. Let P_1, P_2 represent two operators respecting the following relations:*

- *$P_1\rho_1 + P_2\rho_2 \in M$, provided that $\rho_1,\rho_2 \in M$.*
- *P_1 is continuous and compact.*
- *P_2 is a contraction mapping.*

It follows that, there exists $u \in M$ satisfying $u = P_1u + P_2u$.

Theorem 2. *Purpose $S : \mathcal{J} \times \mathbb{R}^4 \to \mathbb{R}$ is continuous and holds for (A_1) defined above. Further, let*

(A_2) $\|S(t,\rho)\| \leq \varpi(t), \quad \forall (t,\rho) \in J \times \mathbb{R}^4$ and $\varpi \in C([0,\beta],\mathbb{R}_+)$.

This implies that Eq. (2.1) will have at least one solution for every $\Xi_K\|\rho_1(t_0) - \rho_2(t_0)\| < 1$.

Proof. Letting $\sup_{t\in J}|\varpi(t)| = \|\varpi\|$ and $\chi \geq \|\rho_0\| + \Theta\|\varpi\|$, we consider $\mathbf{B}_\chi = \{\rho \in \mathbb{E} : \|\rho\| \leq \chi\}$. Suppose P_1, P_2 operators on $\mathbf{B}_\chi$ are written as

$$(P_1\rho)(t) = \frac{1}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}S(\mu,\rho(\mu))d\mu \quad t \in J,$$

and

$$(P_2\rho)(t) = \rho(t_0), \quad t \in J.$$

Now, each $\rho_1,\rho_2 \in \mathbf{B}_\chi$, produces the following:

$$\begin{aligned} \|(P_1\rho_1)(t) + (P_2\rho_2)(t)\| &\leq \|\rho_0\| + \frac{1}{\Gamma(r)}\int_0^t (t-\mu)^{r-1}\|S(\mu,\rho_1(\mu))\|d\mu \\ &\leq \|\rho_0\| + \Theta\|\varpi\| \\ &\leq \chi < \infty. \end{aligned} \tag{4.9}$$

□

Thus, $P_1\rho_1 + P_2\rho_2 \in \mathbf{B}_\chi$. In the forthcoming steps, the contraction of P_2 will be proved in an step by step manner. In this regard, for $t \in J$ and $\rho_1,\rho_2 \in \mathbf{B}_\chi$, we reach at the following inequality:

$$\|(P_1\rho_1)(t) - (P_2\rho_2)(t)\| \leq \|\rho_1(t_0) - \rho_2(t_0)\|. \tag{4.10}$$

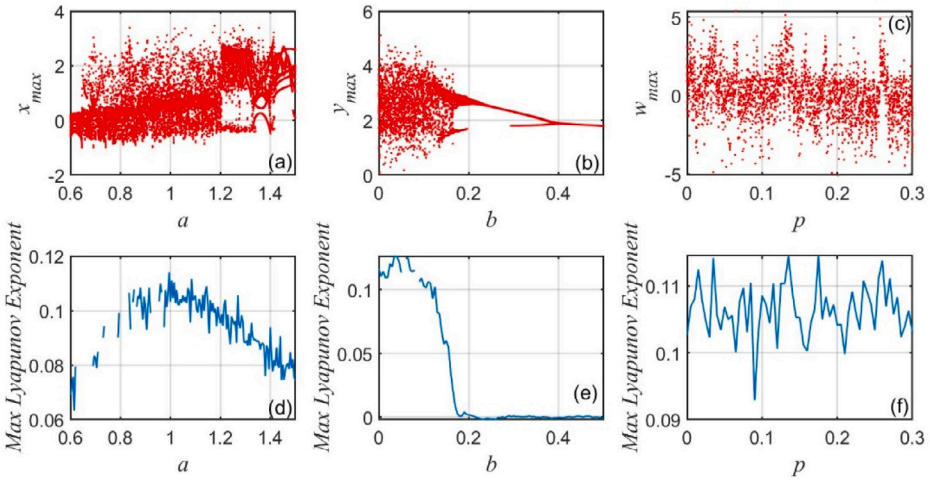


Fig. 2. Bifurcation illustration and max Lyapunov exponents of system (3.20).

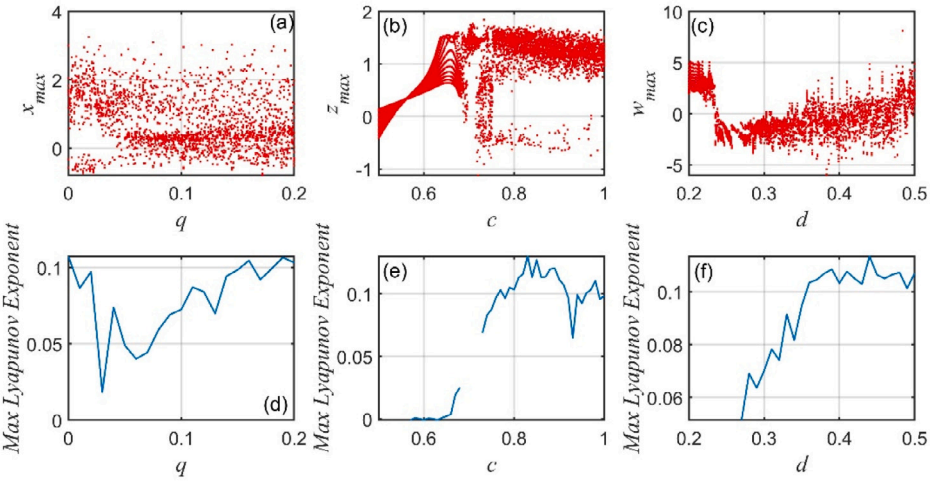


Fig. 3. Bifurcation illustration and max Lyapunov exponents of system (3.20).

Here P_1 will also be continuous since S serves as a continuous function. In addition, for every $t \in J$ and $\rho_1 \in \mathbf{B}_\chi$,

$$\|P_1\rho\| \leq \Theta\|\varpi\| < +\infty,$$

implies that P_1 is uniformly bounded. It can finally be justified that P_1 is compact. Letting $\sup_{(t,\rho)\in J\times\mathbf{B}_\chi} \|S(t,\rho(t))\| = S^*$, it yields the following:

$$\begin{aligned}\|(P_1\rho)(t_1) - (P_2\rho)(t_2)\| &= \frac{1}{\Gamma(r)}\left\| \int_0^{t_1} [(t_2-\mu)^{r-1} - (t_1-\mu)^{r-1}] S(\mu,\rho(\mu))d\mu \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-\mu)^{r-1} S(\mu,\rho(\mu))d\mu \right\| \\ &\leq \frac{S^*}{\Gamma(r)}\left[2(t_2-t_1)^r + (t_2^r - t_1^r)\right] \to 0, \text{ as } t_2 \to t_1. \end{aligned} \tag{4.11}$$

Based on the principle of Arzelá Ascoli, the above-obtained result proves that the Caputo fractional-order model (2.1) has at least one solution.

4.2. Dynamics of new hyperchaotic finance system with variation in parameters a, b, c, d, p, and q

The numerical solutions of system (3.20) are obtained, and the sensitivity analyses of parameters a, b, and p are carried out to examine the influence in relation to the hyperchaotic finance (HF) system (3.20). Take the parameter values $a = 1$, $b = 0.1$, $c = 0.8$, $d = 0.4$, $p = 0.3$, $q = 0.2$, $r = 0.95$ and initial state $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, the bifurcation diagrams and max Lyapunov exponent are drawn in Figs. 2 and 3 to show the rich dynamic behaviors a, b, c, d, p and q change, a varies from 0.6 to 1.5, b varies from 0 to 0.5, c varies 0.5 to 1, d varies 0.2 to 0.5, p varies from 0 to 0.3, q varies 0 to 0.2. Figs. 4 and 5 show the C0 and SE Complexity of the financial hyperchaotic system (3.20).

To examine the effects of "a" on dynamic behaviors of the system (3.20), the study considers the following values for $b = 0.1, c = 0.8, d = 0.4, p = 0.3, q = 0.2, r = 0.95$, and initial values $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, and letting a to be our critical variable. Fig. 6 shows the phase portrait and p-s graphs for 0–1 test of the financial hyperchaotic system (3.20) using selected values of a.

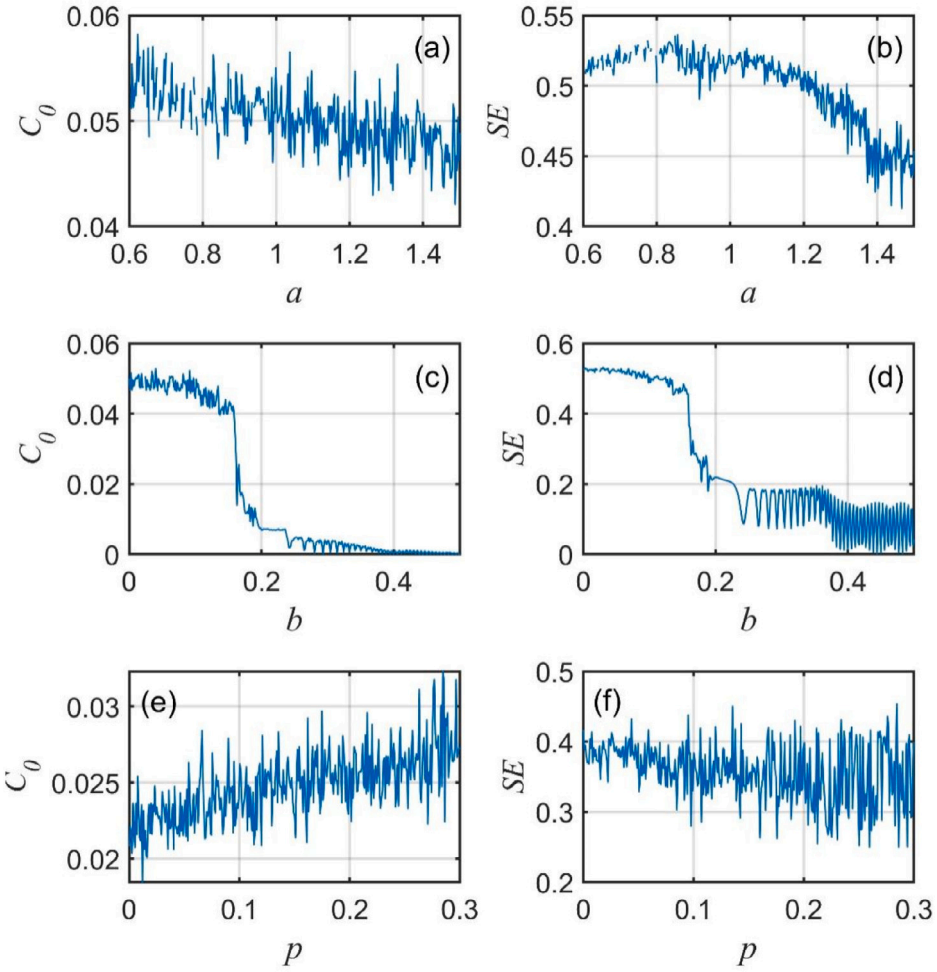


Fig. 4. C_0 and SE Complexity of financial hyperchaotic system (3.20).

To examine the effects of "b" on dynamic behaviors of system (3.20), the study considers the following values for $a = 1$, $c = 0.8$, $d = 0.4$, $p = 0.3$, $q = 0.2$, $r = 0.95$, and initial values $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, and letting "$b$" to be our critical variable.

Fig. 7 shows the phase portrait and *p-s* graphs for 0–1 test of the system (3.20) using selected values of b. The results show that with $b = 0.1$, the scheme (3.20) is found to be in chaotic state but as b rises to 0.3 and 0.5, the system now said to be in periodic state, implying that its stability improve as the value of b increases.

To examine the effects of "c" on dynamic behaviors of system (3.20), the study considers the following values for $a = 1$, $b = 0.1$, $d = 0.4$, $p = 0.3$, $q = 0.2$, $r = 0.95$, and initial values $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, and letting "$c$" to be our critical variable. Fig. 8 shows the phase portrait and *p-s* graphs for 0–1 test of the financial hyperchaotic system (3.20) using selected values of c.

To examine the effects of "d" on dynamic behaviors of system (3.20), the study considers the following values for $a = 1$, $b = 0.1$, $c = 0.8$, $p = 0.3$, $q = 0.2$, $r = 0.95$, and initial values $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, and letting "$d$" to be our critical variable. Fig. 9 shows the phase portrait and *p-s* graphs for 0–1 test of the financial hyperchaotic system (3.20) using selected values of d.

To examine the effects of "p" on dynamic behaviors of system (3.20), the study considers the following values for $a = 1$, $b = 0.1$, $c = 0.8$, $d = 0.4$, $q = 0.2$, $r = 0.95$, and initial values $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, and letting "$p$" to be our critical variable. Fig. 10 shows the phase portrait and *p-s* graphs for 0–1 test of the financial hyperchaotic system (3.20) using selected values of p.

To examine the effects of "q" on dynamic behaviors of system (3.20), the study considers the following values for $a = 1$, $b = 0.1$, $c = 0.8$, $d = 0.4$, $p = 0.3$, $r = 0.95$, and initial values $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, and letting "$q$" to be our critical variable. Fig. 11 shows the phase portrait and *p-s* graphs for 0–1 test of the financial hyperchaotic system (3.20) using selected values of q.

4.3. Dynamics of hyperchaotic finance system with the variation in fractional order

To investigate the effect of order "r" on dynamic behavior of system (3.20), the study considers the following values for $a = 1$, $b = 0.3$, $c = 0.8$, $d = 0.4$, $p = 0.3$, $q = 0.2$ with the initial state $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, and fixed "$r$" as the critical variable.

Fig. 12 depicts the bifurcation plot, demonstrating the complex dynamic behavior as order "r" changes and includes chaotic and periodic motions. The bifurcation plot, which is depicted in Fig. 12, is also compatible with the max Lyapunov exponent and Fig. 13 illustrates the C0 and SE Complexity of system (3.20). Additionally, Fig. 14 presents the contour plots of complexity in the *r-b* plane. Figs. 15 and 16 provide more visual representations of the phases of the system and time series plots of *x, y, z,* and *w* state variables with various orders, respectively.

5. Control of chaotic generalized fractional systems (GFSs)

This section introduces the control for fractional-order hyperchaotic finance system using Laplace transform and the theorem stated below:

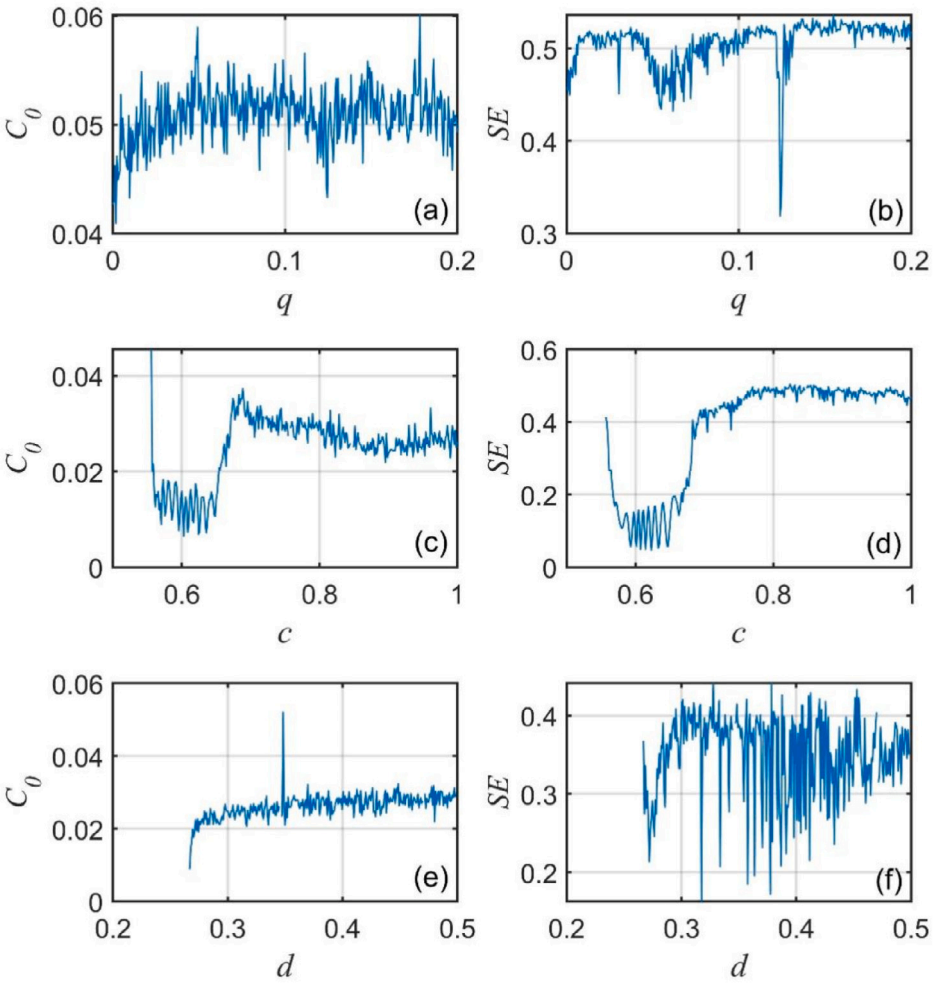


Fig. 5. C_0 and SE Complexity of financial hyperchaotic system (3.20).

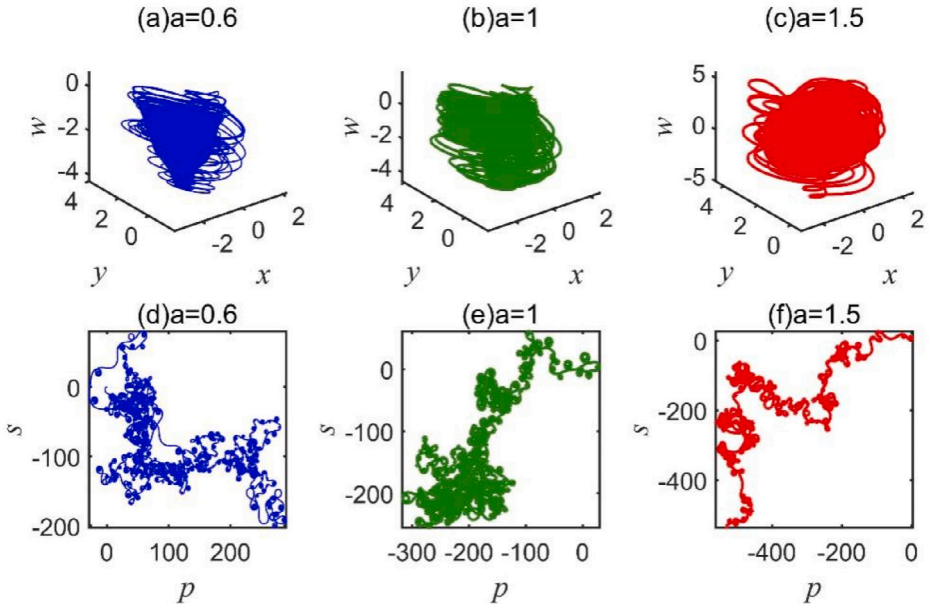


Fig. 6. Phase portrait and *p-s* plot of 0–1 testing for system (3.20).

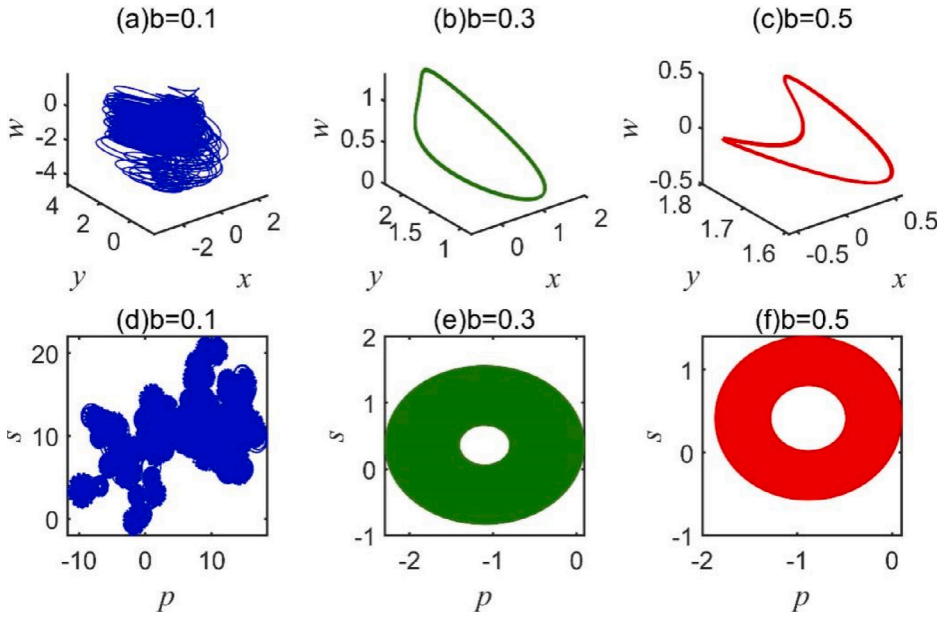


Fig. 7. Phase portrait and *p-s* plot of 0–1 test for system (3.20).

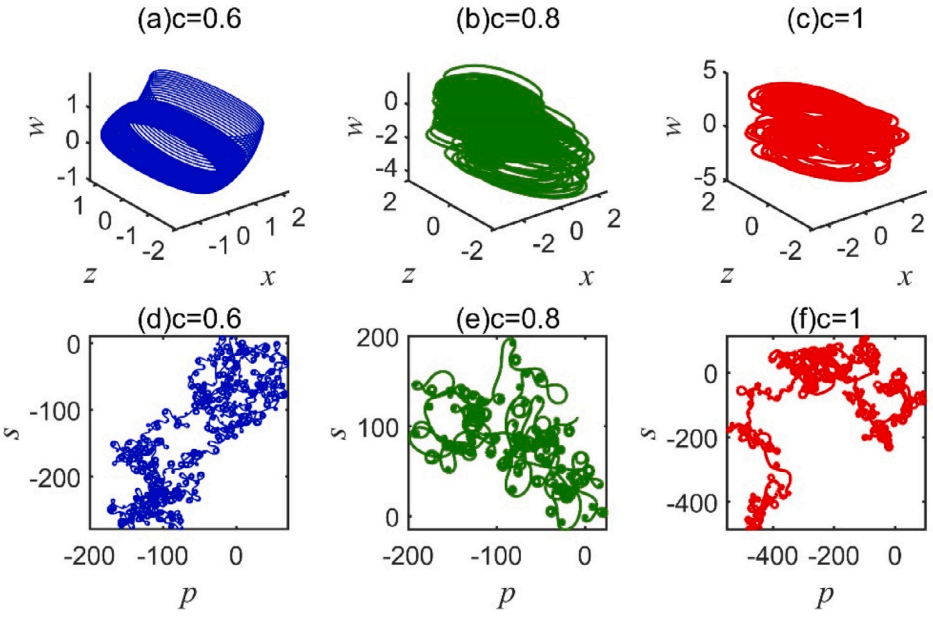


Fig. 8. Phase portrait and *p-s* plot of 0–1 test for system (3.20).

Theorem 3 ([48]). *Let* $F(s) = \mathcal{L}\{f(t)\}$. *If every pole of* $sF(s)$ *is in the complex left-half open plane, we have* $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

Our study focuses on using linear feedback control to regulate solutions of the hyperchaotic finance system of fractional-order. For the choice $a = 1$, $b = 0.1$, $c = 0.8$, $d = 0.4$, $p = 0.3$, $q = 0.2$, $r = 0.95$, and initial state $(x_0, y_0, z_0, w_o) = (0.6, 0.4, 0.6, 0.4)$, we proved that system (3.20) has hyperchaotic solution. By adding control functions to the fractional-order hyperchaotic finance system (3.20), it becomes

$$\begin{cases} D^r x = (y-a)\,x + z + qw - k_1 x, \\ D^r y = 1 - by - x^2 - k_2 y, \\ D^r z = -x - cz - k_3 z, \\ D^r w = -dxy - pz - k_4 w. \end{cases} \tag{5.1}$$

with the control gain matrix given as $K = diag\,(k_1,\ k_2,\ k_3,\ k_4)$.

Theorem 4. *System (5.1) can be approach to fixed point if the control functions take the form:* $k_1 \in R/\{-a\}$, $k_2 \in R/\{-b\}$, $k_3 \in R/\{-c\}$ *and* $k_4 \in R/\{0\}$.

Proof. Using the Laplace transform for system (5.1), we get:

$$\begin{cases} s^r X(s) - x(0) = L\{x(t)\,y(t)\} - aX(s) + Z(s) + qW(s) - k_1 X(s), \\ s^r Y(s) - y(0) = \frac{1}{s} - bY(s) - Lx^2 - k_2 Y(s), \\ s^r Z(s) - z(0) = -X(s) - cZ(s) - k_3 Z(s), \\ s^r W(s) - W(0) = -dLxy - pZ(s) - k_4 W(s). \end{cases} \tag{5.2}$$

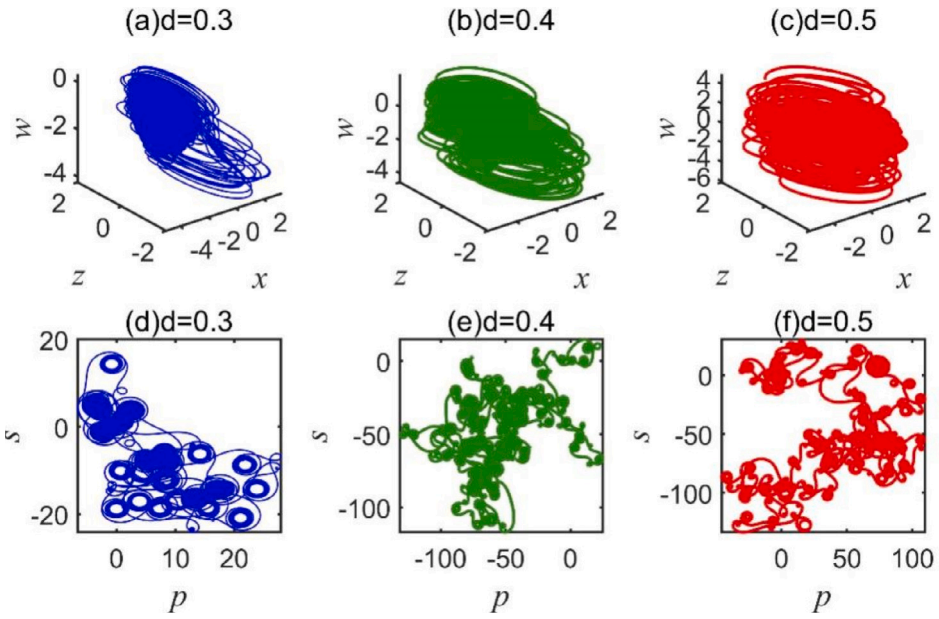


Fig. 9. Phase portrait and the *p-s* plot of the 0–1 test for system (3.20).

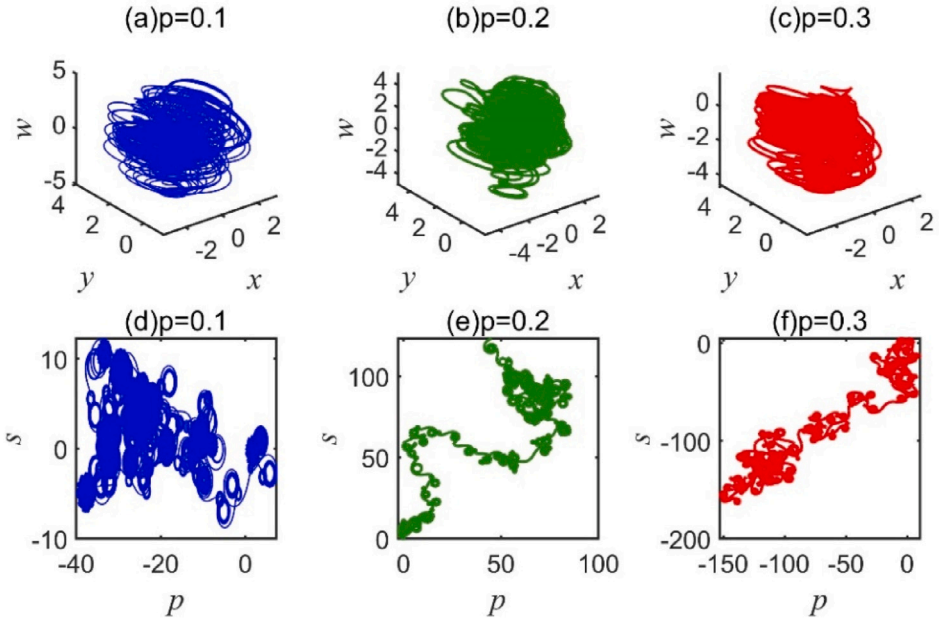


Fig. 10. Phase portrait and the *p-s* plot of the 0–1 test for system (3.20).

Then,

$$\begin{cases} X(s) = \frac{1}{s^{\alpha}+a+k_1}(x(0) + L\{x(t)\,y(t)\} + Z(s) + qW(s)), \\ Y(s) = \frac{1}{s^{\alpha}+b+k_2}(y(0) + \frac{1}{s} - Lx^2), \\ Z(s) = \frac{1}{s^{\alpha}+c+k_3}(z(0) - X(s)), \\ W(s) = \frac{1}{s^{\alpha}+k_4}(w(0) - dL\{xy\} - pZ(s)). \end{cases} \tag{5.3}$$

Using Theorem 4 and the Eq. (5.1), we obtain:

$$\lim_{t\to\infty}(x(t),\ y(t),\ z(t),\ w(t)) = \left(0, \frac{1}{b+k_2},\ 0,\ 0\right). \tag{5.4}$$

It notes that the solution of system (5.1) approaches fixed point after adding the control functions, i.e, the proof of Theorem 4. is completed. □

5.1. Numerical simulation

This subsection contains the numerical simulation for the fractional-order hyperchaotic finance system after adding the control system (5.1). The Predictor-Corrected method is used to find the numerical solution for system (5.1). We choose $K = \text{diag}(1,\ 1,\ 2,\ 1)$ while maintaining parameter values and initial points of (3.20). The solution goes to fixed point $E_0 = \left(0, \frac{1}{b+1},\ 0,\ 0\right)^T$, as shown in Figs. 17 and 18.

6. Conclusions

This paper employs the ADM to address the solution of a financial system with a fractional-order component incorporating profit margin. Through numerical simulations and analyses, including the assessment

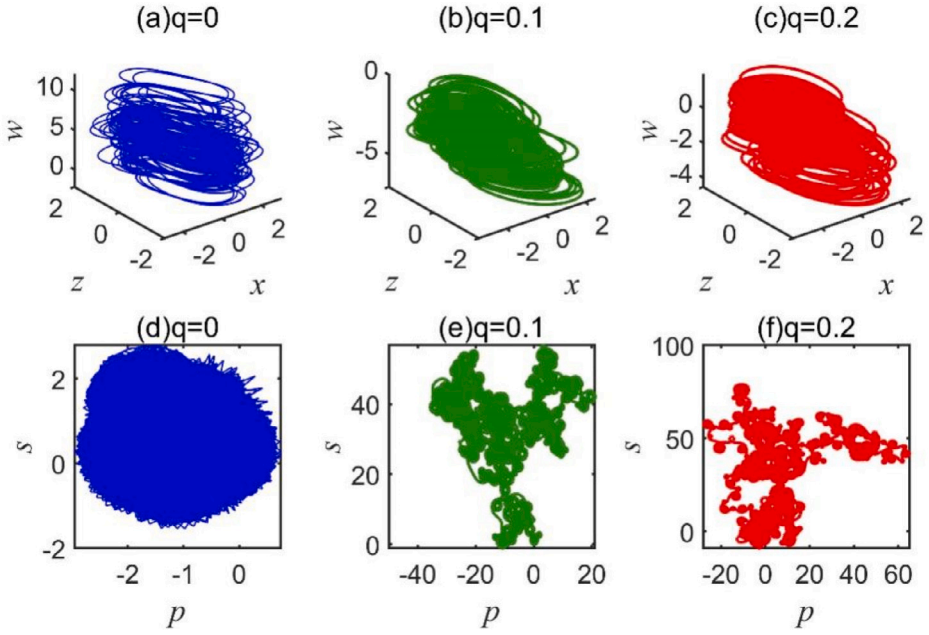


Fig. 11. Phase portrait and the *p-s* plot of the 0–1 test for system (3.20).

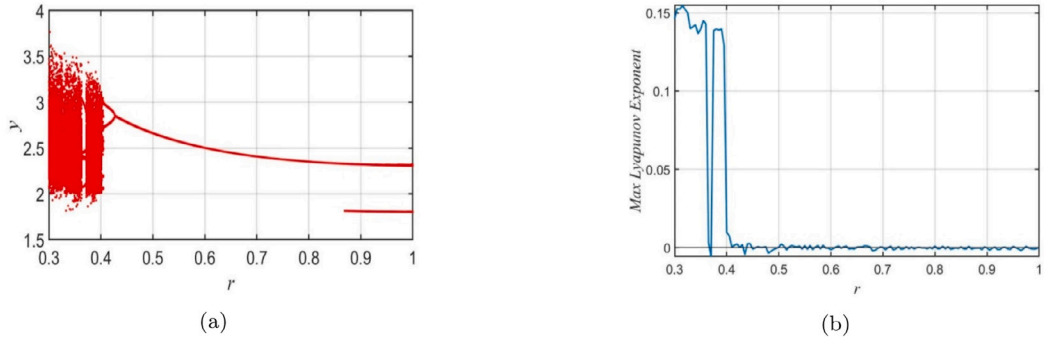


Fig. 12. Matlab simulation of system (3.20) for order r (a) bifurcation illustration and (b) Maximum Lyapunov exponent.

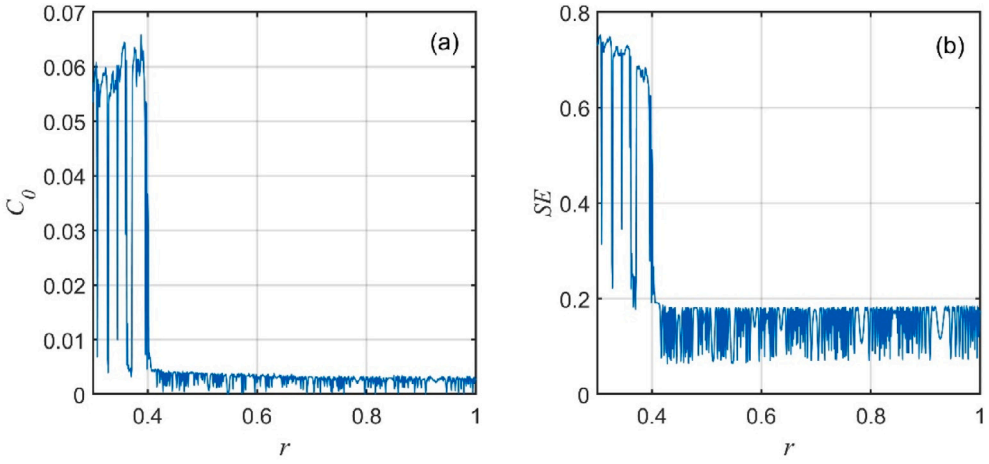


Fig. 13. Complexity of system (3.20) with order r varying from 0.3 to 1.

of Lyapunov exponents, the construction of bifurcation diagrams, complexity analysis, and the 0–1 test, a comprehensive understanding of the system is obtained. Our findings reveal the manifestation of periodic, chaotic, and hyperchaotic dynamics in the system. Furthermore, the overall criteria for the existence and the uniqueness of the exact solutions for Caputo fractional-order models are presented. Additionally, we introduce a control methodology for the fractional-order hyperchaotic financial system using the Laplace transform and linear feedback control. Notably, the research establishes a significant correlation between analytical results and numerical simulations,

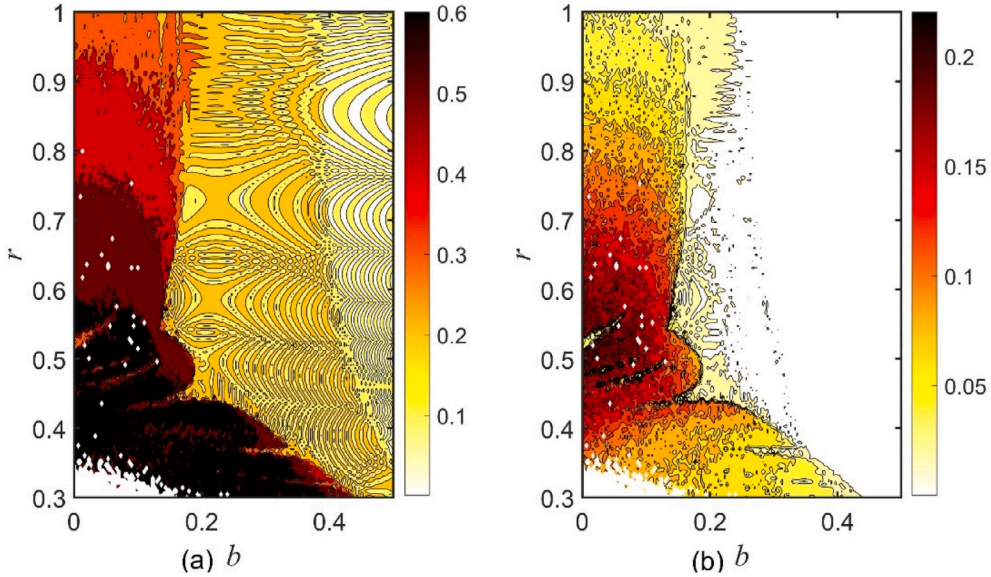


Fig. 14. Complexity contour plots of system (3.20) in the *r*-*b* plane.

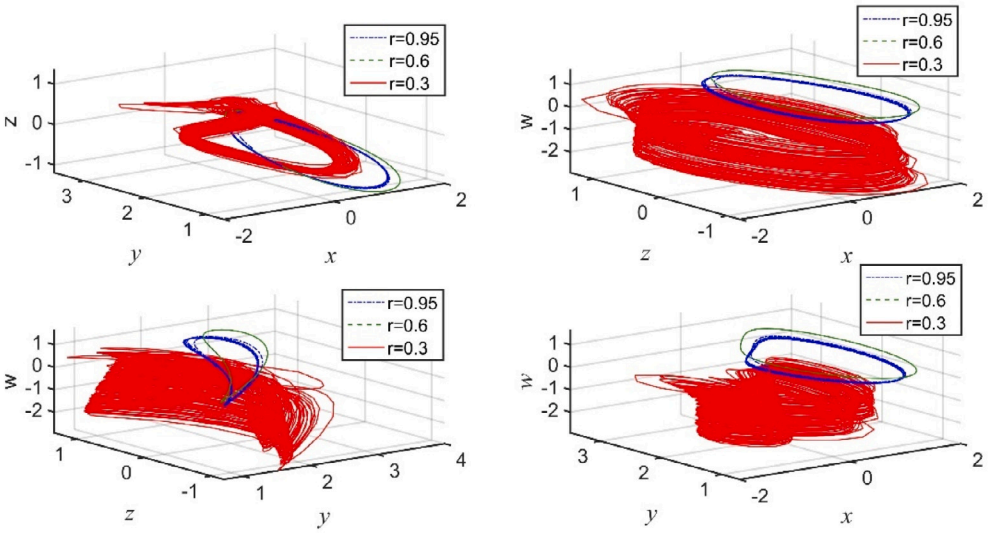


Fig. 15. Phase portraits of system (3.20) using various orders.

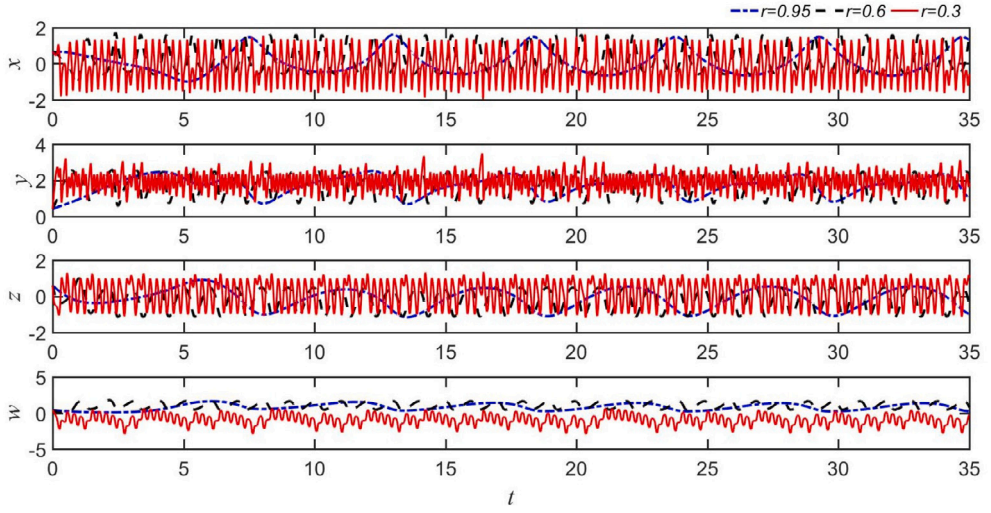


Fig. 16. Time series of *x*, *y*, *z*, and *w* of system (3.20) using various orders.

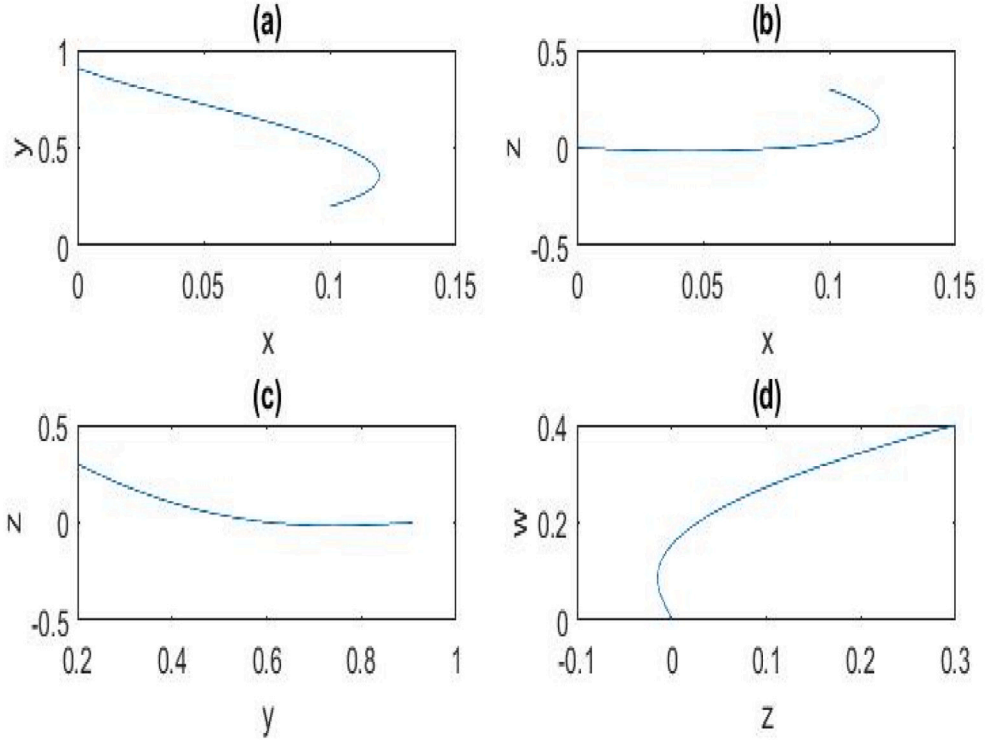


Fig. 17. The solution of hyperchaotic finance system of fractional-order with control Eq. (5.1): (a) (x-y) space, (b) (x-z) space, (c) (y-z) space, and (d) (z-w) space.

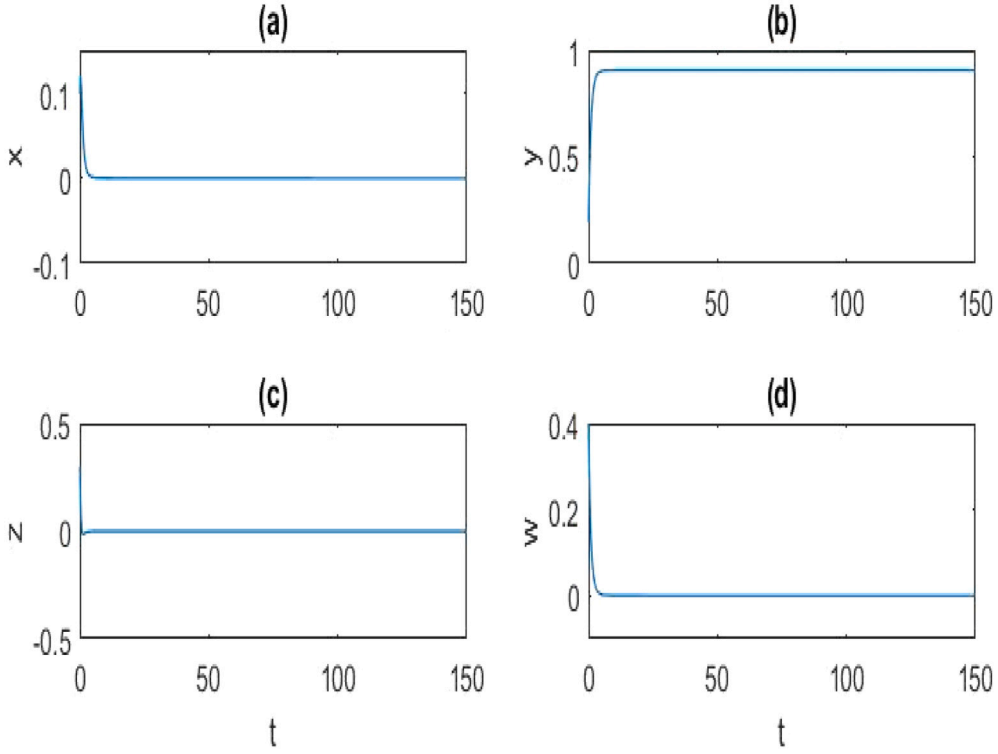


Fig. 18. The solution of hyperchaotic finance scheme of fractional-order with control Eq. (5.1): (a) (t, x) plot, (b) (t, y) plot, (c) (t, z) plot, and (d) (t, w) plot.

underscoring the robustness and efficacy of our methodology. In the future research, we will consider the impact of external factors or exogenous variables on the fractional-order hyperchaotic financial system. This could involve analyzing the system's response to external shocks or changes in economic conditions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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